Lecture 10: Counting Zeroes and Poles

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Meromorphic functions

Definition

A function f is meromorphic on U if it is analytic on U except at a discrete set of points $\{z_j\}$, and each z_j is a pole.

- f, g meromorphic on $U \;\Rightarrow\; \dfrac{f(z)}{g(z)}$ is meromorphic on U.

Poles of $\dfrac{f(z)}{g(z)}$ can occur at: the poles of f and the zeroes of g.

- Poles of $\dfrac{f'(z)}{f(z)}$ do occur at: the poles of f and the zeroes of f,

$$\text{At a zero } z_j \text{ of order } m_j : \; \mathrm{Res}\left(\frac{f'}{f}, z_j\right) = m_j$$

$$\text{At a pole } w_k \text{ of order } n_k : \; \mathrm{Res}\left(\frac{f'}{f}, w_k\right) = -n_k$$

Counting Zeroes and Poles

Theorem

Suppose f is meromorphic on E, Γ a cycle in E with $\mathrm{ind}_\Gamma(z) = 0$ for $z \notin E$, and f has no zeroes or poles on Γ. Let $\{z_j\}$ be the zeroes of f, with respective orders m_j, and $\{w_k\}$ the poles of f, with respective orders n_k. Then

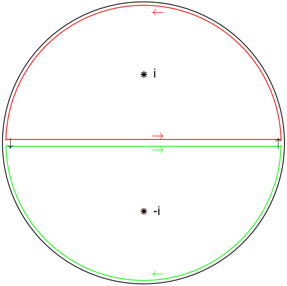
$$\int_\Gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i\left(\sum_j \mathrm{ind}_\Gamma(z_j)\cdot m_j - \sum_k \mathrm{ind}_\Gamma(w_k)\cdot n_k\right)$$

Proof. Follows by the Residue Theorem: the poles of $\dfrac{f'(z)}{f(z)}$ coincide with the combined collection of zeroes and poles of f.

Example. Let $f(z) = \dfrac{z+i}{z-i}$: $\begin{cases} \text{pole of order 1 at } z = i, \\ \text{zero of order 1 at } z = -i. \end{cases}$

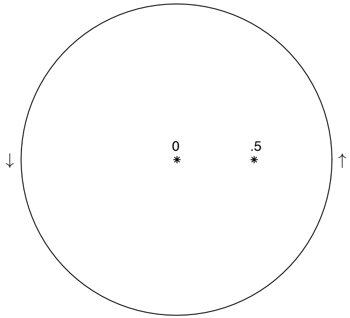
$$\int_{|z|=2} \frac{f'(z)}{f(z)}\, dz = 0$$

- $\Gamma = [-2,2] + \{2e^{it} : t \in [0,\pi]\}$: $\displaystyle\int_\Gamma \frac{f'(z)}{f(z)}\, dz = -2\pi i$
- $\Gamma = [-2,2] + \{2e^{-it} : t \in [0,\pi]\}$: $\displaystyle\int_\Gamma \frac{f'(z)}{f(z)}\, dz = -2\pi i$



Example. Let $f(z) = \dfrac{z^n}{z - \frac{1}{2}}$: $\begin{cases} \text{pole of order 1 at } z = \frac{1}{2}, \\ \text{zero of order n at } z = 0. \end{cases}$

$$\int_{|z|=1} \frac{f'(z)}{f(z)}\, dz = 2\pi i(n-1)$$



Interpretation in terms of $f \circ \gamma$

If γ is a path in E, f analytic on E, can form image path $f \circ \gamma$

$$(f \circ \gamma)(t) = f(\gamma(t)), \qquad \text{so} \qquad (f \circ \gamma)'(t) = f'(\gamma(t))\,\gamma'(t)$$

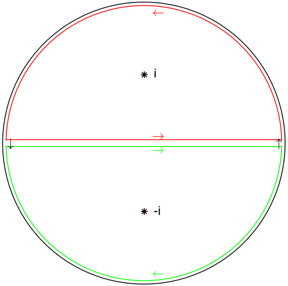
For any path γ:

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz \stackrel{\text{def}}{=} \int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\,\gamma'(t)\,dt = \int_a^b \frac{(f \circ \gamma)'(t)}{(f \circ \gamma)(t)}\,dt = \int_{f \circ \gamma} \frac{1}{z}\,dz$$

Theorem

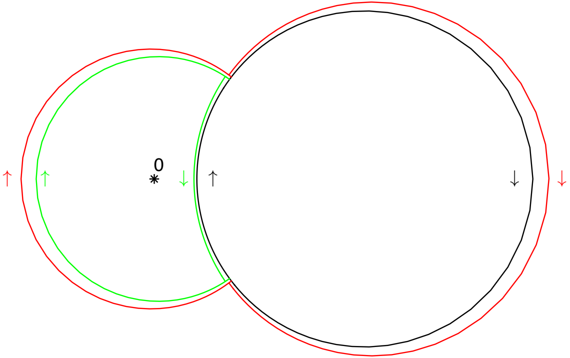
Suppose f is meromorphic on E, Γ a cycle in E with $\operatorname{ind}_\Gamma(z) = 0$ for $z \notin E$, and f has no zeroes or poles on Γ. Let $\{z_j\}$ be the zeroes of f, with respective orders m_j, and $\{w_k\}$ the poles of f, with respective orders n_k. Then

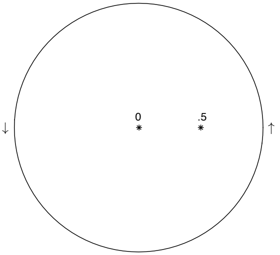
$$\left(\sum_j \operatorname{ind}_\Gamma(z_j) \cdot m_j - \sum_k \operatorname{ind}_\Gamma(w_k) \cdot n_k \right) = \operatorname{ind}_{f \circ \Gamma}(0)$$



$$f(z) = \frac{z+i}{z-i}$$

Image $f \circ \gamma$





$$f(z) = \frac{z^3}{z - \frac{1}{2}}$$

Image $f \circ \gamma$