Lecture 11: Rouche's Theorem

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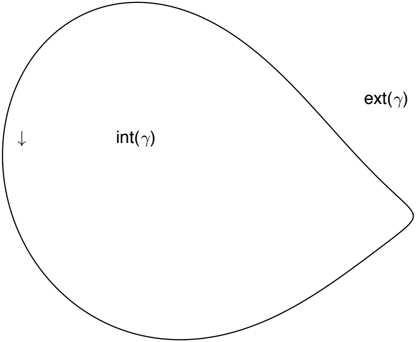
Math 428, Winter 2020

Notation and Definitions

- A **simple path** is a closed path γ such that $\mathbb{C} \setminus \{\gamma\}$ has exactly 2 components, which we call $\text{int}(\gamma)$ and $\text{ext}(\gamma)$, such that: $$\text{ind}_\gamma(z) = \begin{cases} 1\,, & z \in \text{int}(\gamma) \\ 0\,, & z \in \text{ext}(\gamma) \end{cases}$$
- Then: $\text{ind}_\gamma(z) = 0$ for $z \notin E \;\Leftrightarrow\; \text{int}(\gamma) \subset E$.



Theorem

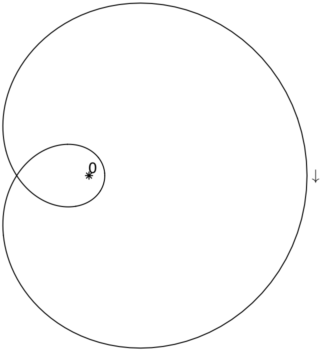
For h meromorphic on E, γ a simple path in E with $\mathrm{int}(\gamma) \subset E$, and h has no zeroes or poles on γ: If the zeroes of h inside γ occur at $\{z_j\}$ with order m_j, and the poles inside γ occur at $\{w_k\}$ with order n_k, then: $\mathrm{ind}_{h\circ\gamma}(0) = \sum_j m_j - \sum_k n_k$.

Example.

$$h(z) = \frac{z}{(z-\frac{1}{2})(z-1)^2}$$

$$\gamma = \partial D_2(0)$$

$h \circ \gamma$:



Observation: If f, g are **analytic** on $E \supset \text{int}(\gamma)$ and $h = \dfrac{f}{g}$,

$$\#\{\text{zeroes of } f \text{ inside } \gamma\} - \#\{\text{zeroes of } g \text{ inside } \gamma\}$$
$$= \#\{\text{zeroes of } h \text{ inside } \gamma\} - \#\{\text{poles of } h \text{ inside } \gamma\}$$

where $\#$ counts the orders of the zeroes (and poles).

Combined with the previous Theorem, this gives:

Theorem

If f, g are analytic on E, γ a simple path in E with $\text{int}(\gamma) \subset E$, and f, g have no zeroes on γ, then if we let $h = \dfrac{f}{g}$,

$$\text{ind}_{h\circ\gamma}(0) = \#\{\text{zeroes of } f \text{ in } \gamma\} - \#\{\text{zeroes of } g \text{ in } \gamma\}.$$

Rouche's Theorem

If f, g are analytic on E, γ a simple path in E with $\mathrm{int}(\gamma) \subset E$,

f, g have no zeroes on γ, and $\left|\dfrac{f(z)}{g(z)} - 1\right| \leq 1$ for all $z \in \{\gamma\}$,

then: $\quad \#\{\text{zeroes of } f \text{ in } \gamma\} = \#\{\text{zeroes of } g \text{ in } \gamma\}$.

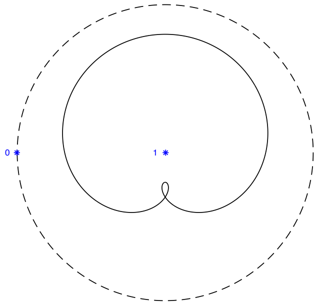
Proof. Let $h(z) = \dfrac{f(z)}{g(z)}$, so $|h(\gamma(t)) - 1| \leq 1$, and $h(\gamma(t)) \neq 0$.

This means $\{h \circ \gamma\} \subset \overline{D_1}(1) \setminus \{0\}$, and thus $\mathrm{ind}(h \circ \gamma)(0) = 0$.

$$f(z) = z^4 - .5z^3 - .3iz^2$$

$$g(z) = z^4$$

$$\gamma = \partial D_1(0)$$



Examples

- $f(z) = \frac{1}{6}z^4 - \frac{1}{2}z^2 + z, \qquad g(z) = z, \qquad \gamma = \partial D_1(0)$
- $f(z) = \frac{1}{6}z^4 - \frac{1}{2}z^2 + z, \qquad g(z) = \frac{1}{6}z^4, \qquad \gamma = \partial D_3(0)$
- $f(z) = 3e^z - z, \qquad g(z) = 3e^z, \qquad \gamma = \partial D_1(0)$
- $f(z) = e^z - 3z, \qquad g(z) = -3z, \qquad \gamma = \partial D_1(0)$
- $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0, \qquad g(z) = z^n,$

  $\gamma = \partial D_R(0)$ where $R > |a_{n-1}| + \cdots + |a_0|$ and $R > 1$.

An n-th order polynomial has exactly n zeroes (incl. multiplicity).