Lecture 13: Inverse Functions II

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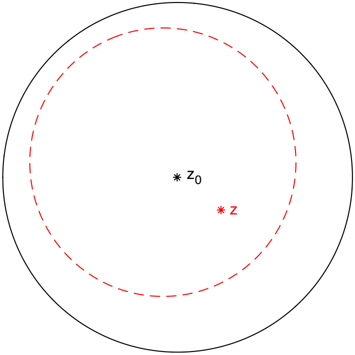
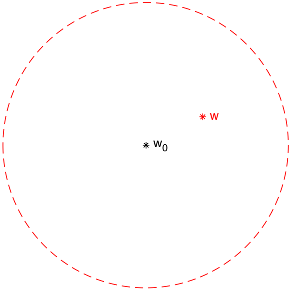
Assume $f(z_0) = w_0$, and $f'(z_0) \neq 0$. Then there exists $f^{-1}(w)$:

- The function $f^{-1}(w)$ is analytic on $D_\delta(w_0)$, some $\delta > 0$.
- $f\big(f^{-1}(w)\big) = w$ for $w \in D_\delta(w_0)$, and $f^{-1}(w_0) = z_0$.
- $f^{-1}(w)' = 1/f'\big(f^{-1}(w)\big)$.



$$f(z) = w \quad \longrightarrow$$



Example

Let $f(z) = \tan z,\ \ z \neq (k+\frac{1}{2})\pi$. Then $f'(z) = (\cos z)^{-2} \neq 0$.

Local inverse for $\tan(z)$ exists at all z_0 in domain of $\tan(z)$.

For any local inverse:

$$(\tan^{-1})'(w) \;=\; \cos^2(\tan^{-1}(w)) \;=\; \frac{1}{1+w^2}$$

Observe: if $|w| < 1$ then $$\frac{1}{1+w^2} = \sum_{k=0}^{\infty}(-1)^k w^{2k}$$

Principal branch of $\tan^{-1}$**:** take $w_0 \;=\; 0,\ \tan^{-1}(0) = 0$. Then

$$\tan^{-1}(w) \;=\; \sum_{k=0}^{\infty}(-1)^k \frac{w^{2k+1}}{2k+1} \quad \text{on } D_\delta(0), \text{ for some } \delta > 0.$$

Claim: $$\tan\left(\sum_{k=0}^{\infty}(-1)^k\frac{w^{2k+1}}{2k+1}\right) = w \quad \text{for all } |w|<1.$$

Proof. The function $\tan\left(\sum_{k=0}^{\infty}(-1)^k\frac{w^{2k+1}}{2k+1}\right) - w$ is analytic on $D_1(0)$, vanishes on $D_\delta(0)$ some $\delta > 0$, so vanishes on $D_1(0)$.

Suppose: $g(w)$ is analytic on a connected open set $E \ni \{0\}$, $g(0) = 0,\ g'(w) = \dfrac{1}{1+w^2}$. **Then**: $\tan\big(g(w)\big) = w,$ all $w \in E$.

Proof. $g(w)$ equals the above series for $\tan^{-1}(w)$ on $D_\delta(0)$, some $\delta > 0$, so $\tan\big(g(w)\big) = w$ on $D_\delta(0)$. E is connected, $\tan\big(g(w)\big)$ is analytic, so $\tan\big(g(w)\big) = w$ on all of E.

An explicit formula for $\tan^{-1}(w)$

$$\tan z = -i\,\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}} = -i\,\frac{e^{2iz}-1}{e^{2iz}+1}$$

Let $u = e^{2iz}$. Then $\tan z = w$ is equivalent to

$$\frac{u-1}{u+1} = iw \qquad \text{which gives} \qquad u = \frac{i-w}{i+w}$$

$$\tan z = w \quad \Leftrightarrow \quad e^{2iz} = \frac{i-w}{i+w} \quad \Leftrightarrow \quad z = \frac{1}{2i}\log\left(\frac{i-w}{i+w}\right)$$

$$\tan^{-1}(w) = \frac{1}{2i}\log\left(\frac{i-w}{i+w}\right)$$

- If $w \notin \{i, -i\}$, there exist infinite many choices for $\tan^{-1}(w)$.
- Values of $\log$ differ by $2\pi k i$; values of $\tan^{-1}(w)$ differ by $k\pi$.