Lecture 14: Simple Connectivity and $\tan^{-1}$

Hart Smith

Department of Mathematics
University of Washington, Seattle

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Theorem

If $E \subset \mathbb{C}$ is open and connected, and $f(z)$ is analytic on E, then $f(z) = F'(z)$ for an analytic F on E if and only if $\int_\gamma f(z)\,dz = 0$ for every closed path γ contained in E.

Proof. $(\Rightarrow)$ If $f = F'$ then $\int_\gamma f(z)\,dz = 0$ by Fund. Thm. Calc.

$(\Leftarrow)$ If $\int_\gamma f(z)\,dz = 0$ all closed γ, we fix $z_0 \in E, c_0 \in \mathbb{C}$, and let

$$F(z) \;=\; c_0 \;+\; \int_{z_0}^{z} f(w)\,dw$$

where $\int_{z_0}^{z}$ denotes integration along **any** path from z_0 to z.

By the proof of Theorem 2.6.1, $F'(z) = f(z)$.

Definition

We say a connected open set $E \subset \mathbb{C}$ is simply connected if, for every closed path γ in E, we have $\text{ind}_\gamma(z) = 0$ for all $z \notin E$.

- That is: a closed path in E can't wind around any $z \notin E$.
- By Cauchy's Theorem, if E is simply connected then for all closed γ in E, all analytic functions f on E, $\displaystyle\int_\gamma f(z)\,dz = 0$

Corollary

If $E \subset \mathbb{C}$ is simply connected, and f is analytic on E, then f has an anti-derivative on E : $f(z) = F'(z)$ for some analytic F on E, and F is determined on E up to a constant.

If we fix $z_0 \in E$, the general anti-derivative F of f is given by

$$F(z) \;=\; c_0 + \int_{z_0}^{z} f(w)\,dw\,, \qquad c_0 \in \mathbb{C}.$$

Example: $E = \mathbb{C} \setminus (-\infty, 0]$ is simply connected.

Proof. We need verify $\text{ind}_\gamma(z) = 0$ for all closed paths γ in E, for all $z \in \mathbb{C} \setminus E,$ that is, for $z \in (-\infty, 0]$.

Key fact: entire half-line $(-\infty, 0]$ lies in unbounded component of $\mathbb{C} \setminus \{\gamma\}$, since it's connected (and unbounded), therefore $\text{ind}_\gamma(z) = 0$ on the entire set $z \in (-\infty, 0]$.

- $f(z) = \dfrac{1}{z}$ has anti-derivative on $\mathbb{C} \setminus (-\infty, 0],\ F(z) = \log z$.

- Principal branch of $\log z$ is the unique anti-derivative of $\dfrac{1}{z}$ on $\mathbb{C} \setminus (-\infty, 0]$ that vanishes at $z = 1$.

Example of interest for $\tan^{-1}(w)$:

$$E = \mathbb{C} \setminus \{ [i, +i\infty) \cup [-i, -i\infty) \}$$

is simply connected.

- There exists an analytic function $g(w)$ on E such that

$$g(0) = 0, \quad g'(w) = \frac{1}{1 + w^2}$$

- By last lecture, $\tan(g(w)) = w$, so $g(w) = \tan^{-1}(w)$.

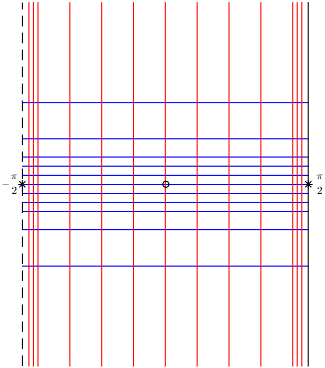
Principal branch of $\tan^{-1}(w)$ is:

$$\tan^{-1}(w) \;=\; \frac{1}{2i}\log\left(\frac{i-w}{i+w}\right) \quad \text{with principal branch of } \log\,.$$

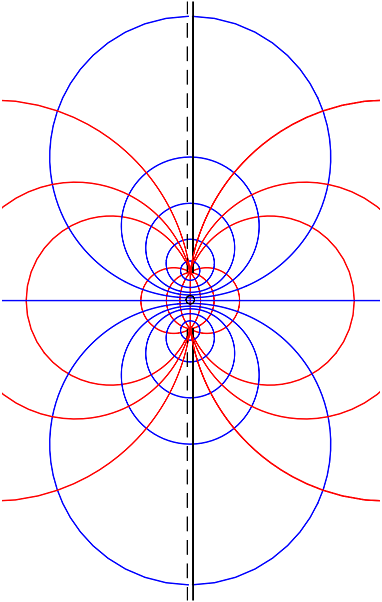
- This is defined on the set $\left\{w \,:\, \dfrac{i-w}{i+w} \notin (-\infty,0]\,,\; w \neq \pm i\right\}$
- The "cut" set is the set of $w \,:\, u = \dfrac{i-w}{i+w} \in (-\infty,0]$.
- Write $w = i\,\dfrac{u-1}{u+1}$. Then $\begin{cases} u \in (-\infty,-1) \,: & w \in (i,+i\infty) \\ u \in (-1,0) \,: & w \in (-i,-i\infty) \end{cases}$

Principal branch of $\tan^{-1}(w)$ maps $\mathbb{C} \setminus \{[i,+i\infty) \cup [-i,-i\infty)\}$ to the vertical strip $-\frac{\pi}{2} < \mathrm{Re}(z) < \frac{\pi}{2}$.

Complex plane representation of $z \to \tan z$



z



$w = \tan z$