Lecture 15: Homotopy of Paths

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Definition: let E be an open set

Suppose that γ_0 and γ_1 are closed paths from $t \in [0,1]$ into E. Then γ_0 is homotopic to γ_1 in E if there is a continuous function

$$h(s,t) \;:\; [0,1] \times [0,1] \to E$$

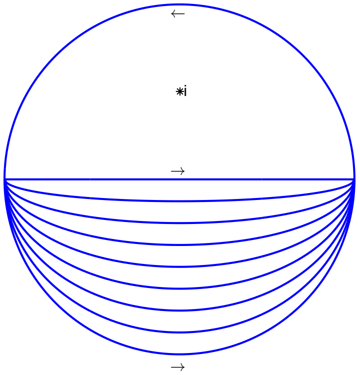
so $h(0,t) = \gamma_0(t)\,,\; h(1,t) = \gamma_1(t)$, and for each $0 \le s \le 1$ the curve $\gamma_s(t) := h(s,t)$ is a closed path in E.

- Loosely speaking: there is a continuous family of paths γ_s that moves from γ_0 to γ_1, and which stays inside E.
- Any two paths in $\mathbb{C}$ are homotopic in $\mathbb{C}$, by letting

$$\gamma_s(t) = (1-s)\gamma_0(t) + s\gamma_1(t)$$

- We say γ_0 is *homotopic to a point* if γ_0 is homotopic to γ_1, where γ_1 is constant (i.e. $\gamma_1(t) = z_0$ for all t, some $z_0 \in E$).

Example: the circle is homotopic to the upper semi-circle in $\mathbb{C} \setminus \{i\}$.



- Writing $h(s,t)$ is a bother (and must use parameter $t \in [0,1]$)

$$h(s,t) = \begin{cases} e^{2\pi i t}, & 0 \le t \le \frac{1}{2}, \\ (1-s)e^{2\pi i t} + s(4t-3), & \frac{1}{2} \le t \le 1. \end{cases}$$

- We rarely actually write out $h(s,t)$.

Topological Definition

A connected open set $E \subset \mathbb{C}$ is simply connected if every closed path in E is homotopic to a point (can be any point in E).

Example: every convex open set is simply connected.

$$h(s,t) \;=\; (1-s)\gamma_0(t) + s\,z_1\,, \quad \text{any } \; z_1 \in E.$$

Definition: a set $E \subset \mathbb{C}$ is star-shaped about the point $z_1 \in E$ if E contains the straight line segment $[z_0, z_1]$ for every $z_0 \in E$.

- A star-shaped domain is simply connected: $h(s,t)$ as above.
- $\mathbb{C} \setminus (-\infty, 0]$ is star-shaped about $z_1 = 1$.
- $\mathbb{C} \setminus \{[i, +i\infty) \cup [-i, -i\infty)$ is star-shaped about $z_1 = 0$.

Theorem 4.6.9

Suppose γ_0 is homotopic to γ_1 in E. Then $\operatorname{ind}_{\gamma_1}(z) = \operatorname{ind}_{\gamma_0}(z)$ for all points $z \in \mathbb{C} \setminus E$.

Proof. If $z \notin E$, then $\operatorname{ind}_{\gamma_s}(z)$ is defined for all $s \in [0,1]$, where $\gamma_s(t) = h(s,t)$ is a homotopy of γ_0 to γ_1. We'll show that

$$\operatorname{ind}_{\gamma_s}(z) \text{ is a continuous function of } s \in [0,1] \text{ if } z \notin E$$

thus it's constant since it can have only integer values. Write

$$\operatorname{ind}_{\gamma_s}(z) = \frac{1}{2\pi i} \int_0^1 \frac{1}{\gamma_s(t) - z} \frac{d\gamma_s(t)}{dt}\, dt$$

by uniform continuity of the integrand in s, the integral is continuous in s.

Theorem 4.6.10

Suppose γ_0 is homotopic to γ_1 in E. If f is analytic on E, then $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_0} f(z)\,dz$. So, if E is simply connected, then $\int_{\gamma} f(z)\,dz = 0$ for every closed path in E, if f is analytic on E.

Proof. Consider the cycle $\Gamma = \gamma_1 - \gamma_0$ in E. Then

$$\operatorname{ind}_\Gamma(z) = \operatorname{ind}_{\gamma_1}(z) - \operatorname{ind}_{\gamma_0}(z) = 0 \text{ if } z \notin E.$$

By the Cauchy Integral Theorem for cycles, if f is analytic on E

$$0 = \int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_0} f(z)\,dz$$

If E is simply connected, take $\gamma_1 = z_1$, note $\int_{\{z_1\}} f(z)\,dz = 0$.

- Topological simple connectivity implies our earlier definition:

$$\text{ind}_\gamma(z) = 0 \text{ for all closed paths in } E \text{, all } z \notin E$$

since $\text{ind}_{\gamma_1}(z) = 0$ if γ_1 is a constant path, and $z \neq z_1$.

- Converse also holds, but is much more difficult to prove.

  What's important for us is that, whichever definition you verify

On a simply connected set E

- Every analytic function has an anti-derivative on E.
- $\displaystyle\int_\gamma f(z)\,dz = 0$ if γ is a closed path in E and f analytic on E.