

Lecture 17: Integrals over $[0, 2\pi]$

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Converting $\int_0^{2\pi}$ to a contour integral

Goal: Evaluate $\int_0^{2\pi} f(e^{it}) dt$ for a function $f(z)$.

Method: Let $z = e^{it}$, $t \in [0, 2\pi]$: $\frac{dz}{z} = \frac{ie^{it}dt}{e^{it}} = i dt$

$$\int_0^{2\pi} f(e^{it}) dt = -i \int_{|z|=1} f(z) \frac{1}{z} dz$$

Example: $\int_0^{2\pi} \frac{1}{2 + e^{it}} dt = -i \int_{|z|=1} \frac{1}{(2+z)z} dz$

By Residue Theorem:

$$\text{integral} = (2\pi i)(-i) \operatorname{Res}\left(\frac{1}{(2+z)z}, 0\right) = \pi$$

Integrals with $\sin(t), \cos(t)$: if $z = e^{it} \Rightarrow z^{-1} = e^{-it}$

$$\int_0^{2\pi} f(\cos(t)) dt = -i \int_{|z|=1} f\left(\frac{z+z^{-1}}{2}\right) \frac{1}{z} dz$$

$$\int_0^{2\pi} f(\sin(t)) dt = -i \int_{|z|=1} f\left(\frac{z-z^{-1}}{2i}\right) \frac{1}{z} dz$$

Example: $\int_0^{2\pi} \frac{1}{3 + \sin(t)} dt = -i \int_{|z|=1} \frac{1}{3 + (z - z^{-1})/2i} \frac{dz}{z}$

$$= 2 \int_{|z|=1} \frac{1}{6iz + z^2 - 1} dz$$

Roots: $z = -3i \pm \sqrt{-8} = -(3 \pm \sqrt{8})i$; $-(3 + \sqrt{8})i \notin D_1(0)$.

Ans $= 4\pi i \operatorname{Res}\left(\frac{1}{(z+(3+\sqrt{8})i)(z+(3-\sqrt{8})i)}, -(3-\sqrt{8})i\right) = \frac{\pi}{\sqrt{2}}$

Works for any rational function of $\sin(t), \cos(t)$

$$\int_0^{2\pi} \frac{1}{(4 + \cos(t))^2} dt = -i \int_{|z|=1} \frac{1}{(4 + (z + z^{-1})/2)^2} \frac{dz}{z}$$

Put in standard quadratic form:

$$-4i \int_{|z|=1} \frac{1}{(8 + z + z^{-1})^2} \frac{dz}{z} = -4i \int_{|z|=1} \frac{z}{(8z + z^2 + 1)^2} dz$$

2 Roots: $-4 + \sqrt{15} \in D_1(0)$, $-4 - \sqrt{15} \notin D_1(0)$.

$$\text{Ans} = 8\pi \operatorname{Res}\left(\frac{z}{(z+4+\sqrt{15})^2(z+4-\sqrt{15})^2}, -4 + \sqrt{15}\right)$$

$$= 8\pi \left(\frac{1}{(2\sqrt{15})^2} - \frac{2(-4+\sqrt{15})}{(2\sqrt{15})^3} \right) = \frac{8\pi}{15\sqrt{15}}$$