Lecture 18: Integrals over $\mathbb{R}$

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Math 428, Winter 2020

Converting $\int_{-\infty}^{\infty}$ to a contour integral

Example: Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+t^2}\,dt = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+t^2}\,dt$

For each finite R, this equals a contour integral

$$\int_{-R}^{R} \frac{1}{1+t^2}\,dt = \int_{[-R,R]} \frac{1}{1+z^2}\,dz$$

Idea: add a contour γ_R such that $[-R,R]+\gamma_R$ is closed, and

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{1}{1+z^2}\,dz = 0$$

Then: $$\int_{-\infty}^{\infty} \frac{1}{1+t^2}\,dt = \lim_{R\to\infty} \int_{[-R,R]+\gamma_R} \frac{1}{1+z^2}\,dz$$

Typically: $\gamma_R = \{Re^{it} : t \in [0,\pi]\}$ or $\gamma_R = \{Re^{-it} : t \in [0,\pi]\}$

In both cases, $\ell(\gamma_R) = \pi R$, so have the bound

$$\left| \int_{\gamma_R} \frac{1}{1+z^2}\, dz \right| \leq \pi R \max_{z \in \{\gamma_R\}} \left| f(z) \right| .$$

Useful fact

If $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, there is some M:

$$\tfrac{1}{2}|z|^n < |P(z)| < \tfrac{3}{2}|z|^n, \quad \text{if} \quad |z| \geq M .$$

Proof. For $|z|$ large: $\left| a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \right| < \frac{1}{2}|z|^n$. $\square$

For either choice of γ_R above, for R large enough:

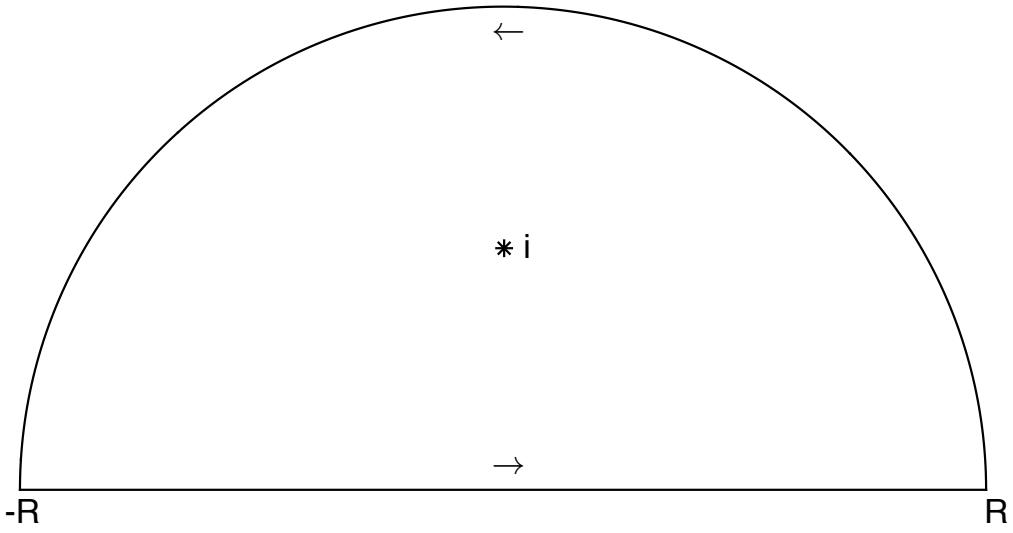
$$\left| \int_{\gamma_R} \frac{1}{1+z^2}\, dz \right| \leq \frac{\pi R}{\frac{1}{2}R^2} \quad \text{so} \quad \lim_{R\to\infty} \int_{\gamma_R} \frac{1}{1+z^2}\, dz = 0$$

For $\frac{1}{1+z^2}$, either upper or lower γ_R work; if choose upper, then

$$\int_{-\infty}^{\infty} \frac{1}{1+t^2}\,dt \;=\; \lim_{R\to\infty} \int_{[-R,R]+\gamma_R} \frac{1}{1+z^2}\,dz$$

but for any $R > 1$,

$$\int_{[-R,R]+\gamma_R} \frac{1}{1+z^2}\,dz \;=\; 2\pi i\,\mathrm{Res}\Big(\frac{1}{1+z^2}\,,i\Big) \;=\; 2\pi i\cdot\frac{1}{2i} \;=\; \pi\,.$$



Lower γ_R gives $-2\pi i\,\mathrm{Res}\Big(\frac{1}{1+z^2}\,,-i\Big) \;=\; -2\pi i\cdot\frac{-1}{2i} \;=\; \pi\,.$

Similar argument shows:

For polynomials $P(z), Q(z)$, if $\text{order}(Q) \geq 2 + \text{order}(P)$, and $Q(z)$ has no zeroes on the real number line, then

$$\int_{-\infty}^{\infty} \frac{P(t)}{Q(t)}\, dt \;=\; 2\pi i \sum_{z_j} \text{Res}\Big(\frac{P}{Q}, z_j\Big) \;=\; -2\pi i \sum_{w_k} \text{Res}\Big(\frac{P}{Q}, w_k\Big)$$

where $\{z_j\}$ are the zeroes of Q in upper half plane $\text{Im}(z) > 0$, and $\{w_k\}$ are the zeroes of Q in the lower half plane $\text{Im}(w) < 0$.

Proof. Key estimate: $\left|\frac{P(z)}{Q(z)}\right| \leq C\, \frac{|z|^{\text{order}(P)}}{|z|^{\text{order}(Q)}} \leq \frac{C}{|z|^2}$ for $|z| > M$,

so
$$\lim_{R\to\infty} \int_{\gamma_R} \frac{P(z)}{Q(z)}\, dz \;=\; 0\,.$$

Example: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(1-it)^2}\, dt \;=\; 0$, since no poles with $\text{Im}(z) > 0$.

Example: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+t^4}\, dt$

Poles of $\dfrac{1}{1+z^4}$ at $z^4 = -1$: $z \in \left\{ \frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}} \right\}$.

In the upper half plane: two poles, both simple. Integral equals:

$$2\pi i \Big(\frac{1}{i\sqrt{2} \cdot \sqrt{2} \cdot (1+i)\sqrt{2}} + \frac{1}{-\sqrt{2} \cdot (-1+i)\sqrt{2} \cdot i\sqrt{2}} \Big)$$

$$= \frac{\pi}{\sqrt{2}} \Big(\frac{1}{1+i} + \frac{1}{1-i} \Big) = \frac{\pi}{\sqrt{2}}$$