

# Lecture 19: Integrals over $\mathbb{R}$ with $e^{\pm it}$

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## Example

Evaluate:  $\int_{-\infty}^{\infty} \frac{e^{it}}{1+t^2} dt = \lim_{R \rightarrow \infty} \int_{[-R,R]} \frac{e^{iz}}{1+z^2} dz$

Want to choose contour  $\gamma_R$  so  $[-R, R] + \gamma_R$  is closed, and

$$\lim_{R \rightarrow \infty} \int_{\gamma_R} \frac{e^{iz}}{1+z^2} dz = 0$$

Then:  $\int_{-\infty}^{\infty} \frac{e^{it}}{1+t^2} dt = \lim_{R \rightarrow \infty} \int_{[-R,R]+\gamma_R} \frac{e^{iz}}{1+z^2} dz$

Consider:  $\gamma_R^+ = \{Re^{it} : t \in [0, \pi]\}$  or  $\gamma_R^- = \{Re^{-it} : t \in [0, \pi]\}$

- For  $z \in \gamma_R^\pm$  :  $\left| \frac{1}{1+z^2} \right| \leq \frac{2}{R^2}$  when  $R$  is large.
- $|e^{iz}| = e^{\operatorname{Re}(iz)} = e^{-\operatorname{Im}(z)}.$
- $z \in \gamma_R^+ \Rightarrow |e^{iz}| \leq 1$ , so  $\left| \int_{\gamma_R^+} \frac{e^{iz}}{1+z^2} dz \right| \leq \pi R \cdot \frac{2}{R^2} \rightarrow 0.$

$$\int_{-\infty}^{\infty} \frac{e^{it}}{1+t^2} dt = 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{1+z^2}, i\right) = \pi e^{-1}$$

Evaluate:  $\int_{-\infty}^{\infty} \frac{\cos t}{1+t^2} dt = \lim_{R \rightarrow \infty} \int_{[-R,R]} \frac{\cos z}{1+z^2} dz$

Write as:  $\frac{1}{2} \lim_{R \rightarrow \infty} \left( \int_{[-R,R]} \frac{e^{iz}}{1+z^2} dz + \int_{[-R,R]} \frac{e^{-iz}}{1+z^2} dz \right)$

- Second term:  $z \in \gamma_R^- \Rightarrow |e^{-iz}| = e^{\operatorname{Im}(z)} \leq 1$

$$\int_{-\infty}^{\infty} \frac{e^{-it}}{1+t^2} dt = -2\pi i \operatorname{Res} \left( \frac{e^{-iz}}{1+z^2}, -i \right) = \pi e^{-1}$$

$$\int_{-\infty}^{\infty} \frac{\cos t}{1+t^2} dt = \pi e^{-1}$$

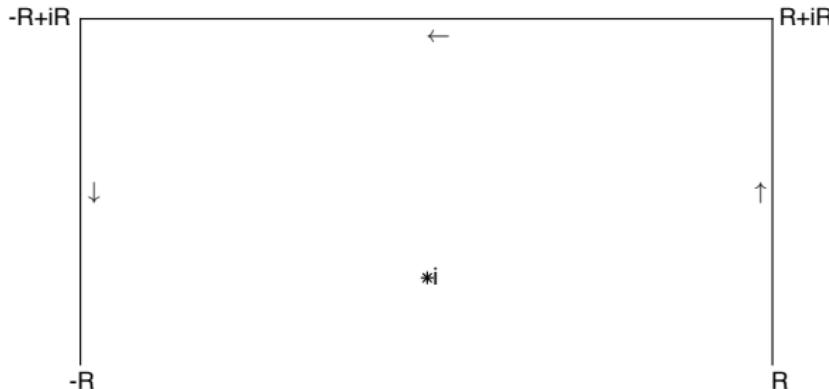
# Integrals that are not absolutely convergent

Evaluate:  $\int_{-\infty}^{\infty} \frac{t \sin t}{1 + t^2} dt = \lim_{R \rightarrow \infty} \int_{[-R, R]} \frac{z \sin z}{1 + z^2} dz$

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To evaluate  $\lim_{R \rightarrow \infty} \int_{[-R, R]} \frac{ze^{iz}}{1 + z^2} dz$  we use the path  $\mu_R^+$

$$\mu_R^+ = [R, R + iR] \cup [R + iR, -R + iR] \cup [-R + iR, -R]$$



**Note:**  $|z| \geq R$  if  $z \in \mu_R^+$ , so:  $\left| \frac{ze^{iz}}{1+z^2} \right| \leq \frac{2e^{-\operatorname{Im}(z)}}{R}$

- $\left| \int_{[R+iR, -R+iR]} \frac{ze^{iz}}{1+z^2} dz \right| \leq 2R \cdot \frac{2e^{-R}}{R} = 4e^{-R} \rightarrow 0$
- Vertical sides require more careful estimate:

$$\int_{[R, R+iR]} \frac{ze^{iz}}{1+z^2} dz = \int_0^R \frac{(R+it)e^{i(R+it)}}{1+(R+it)^2} i dt$$

$$\left| \int_{[R, R+iR]} \frac{ze^{iz}}{1+z^2} dz \right| \leq \int_0^R \left| \frac{(R+it)e^{i(R+it)}}{1+(R+it)^2} \right| dt \leq \int_0^R \frac{2e^{-t}}{R} dt$$

Last term is less than  $2/R$ , so it goes to 0 as  $R \rightarrow \infty$ , thus:

$$\lim_{R \rightarrow \infty} \int_{\mu_R^+} \frac{ze^{iz}}{1+z^2} dz = 0$$

Put this all together:

The integral  $\int_{-\infty}^{\infty} \frac{te^{it}}{1+t^2} dt$  equals:

$$\lim_{R \rightarrow \infty} \int_{[-R,R]+\mu_R^+} \frac{ze^{iz}}{1+z^2} dz = 2\pi i \operatorname{Res}\left(\frac{ze^{iz}}{1+z^2}, i\right) = i\pi e^{-1}$$

If we expand  $e^{it} = \cos t + i \sin t$ , then

$$\left( \int_{-\infty}^{\infty} \frac{t \cos t}{1+t^2} dt \right) + i \left( \int_{-\infty}^{\infty} \frac{t \sin t}{1+t^2} dt \right) = i\pi e^{-1}$$

so:

$$\int_{-\infty}^{\infty} \frac{t \cos t}{1+t^2} dt = 0, \quad \int_{-\infty}^{\infty} \frac{t \sin t}{1+t^2} dt = \pi e^{-1}$$

## Theorem

For polynomials  $P(z)$ ,  $Q(z)$ , if  $\text{order}(Q) \geq 1 + \text{order}(P)$ , and  $Q(z)$  has no zeroes on the real number line, then

$$\int_{-\infty}^{\infty} e^{it} \frac{P(t)}{Q(t)} dt = 2\pi i \sum_{z_j} \text{Res}\left(e^{iz} \frac{P(z)}{Q(z)}, z_j\right)$$

with  $\{z_j\}$  the zeroes of  $Q$  in upper half plane  $\text{Im}(z) > 0$ . Also,

$$\int_{-\infty}^{\infty} e^{-it} \frac{P(t)}{Q(t)} dt = -2\pi i \sum_{w_k} \text{Res}\left(e^{-iz} \frac{P(z)}{Q(z)}, w_k\right)$$

where  $\{w_k\}$  are the zeroes of  $Q$  in lower half plane  $\text{Im}(w) < 0$ .

### Example:

$$\int_{-\infty}^{\infty} \frac{e^{it}}{t+i} dt = 0, \quad \int_{-\infty}^{\infty} \frac{e^{-it}}{t+i} dt = -2\pi i e^{-1}.$$