

Lecture 20: Removable singularities, Fourier transforms

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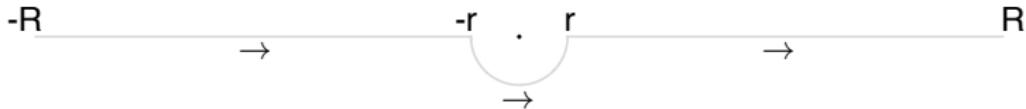
Math 428, Winter 2020

Integrating through a removable singularity

Evaluate: $\int_{-\infty}^{\infty} \frac{\sin t}{t} dt = \lim_{R \rightarrow \infty} \int_{[-R, R]} \frac{\sin z}{z} dz$

- $\frac{\sin z}{z}$ has no poles, but $\frac{e^{iz}}{z}$ and $\frac{e^{-iz}}{z}$ have poles at 0, and we evaluate those terms with different contours.

Idea: since $\frac{\sin z}{z}$ is entire, we can deform $[-R, R]$ to a contour that “goes around 0” but gives the same value for the integral:



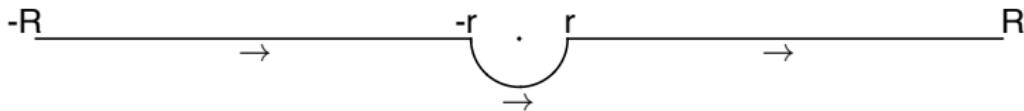
Notation: $[-R, R]_- = [-R, -r] - \gamma_r^- + [r, R]$

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$$\int_{[-R,R]} \frac{\sin z}{z} dz = \int_{[-R,R]_-} \frac{\sin z}{z} dz = \int_{[-R,R]_-} \frac{e^{iz}}{2iz} dz - \int_{[-R,R]_-} \frac{e^{-iz}}{2iz} dz$$

$$\lim_{R \rightarrow \infty} \int_{[-R,R]_-} \frac{e^{iz}}{2iz} dz = \int_{\mu_R^+ + [-R,R]_-} \frac{e^{iz}}{2iz} dz = 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{2iz}, 0\right) = \pi$$

$$\lim_{R \rightarrow \infty} \int_{[-R,R]_-} \frac{e^{-iz}}{2iz} dz = \int_{\mu_R^- + [-R,R]_-} \frac{e^{-iz}}{2iz} dz = 0 \quad (\text{no poles inside})$$

Conclude: $\lim_{R \rightarrow \infty} \int_{-R}^R \frac{\sin t}{t} dt = \pi$

Fourier transform

Definition

If $f(t)$ is a function of $t \in \mathbb{R}$, then for $s \in \mathbb{R}$ define

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-ist} f(t) dt$$

for the values of $s \in \mathbb{R}$ at which the integral exists.

- If $\int |f(t)| dt$ converges then $\hat{f}(s)$ exists for all $s \in \mathbb{R}$.
- $\hat{f}(0) = \int_{-\infty}^{\infty} f(t) dt$ (if the integral converges).
- Are interesting examples where $\hat{f}(s)$ exists except at a finite set of points, and $\hat{f}(s)$ has jumps at those points:

$$f(t) = \frac{\sin t}{t} : \text{jumps at } s = \pm 1, \quad f(t) = \frac{1}{t \pm i} : \text{jump at } s = 0.$$

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Evaluate: $\hat{f}(s)$ for $s \in \mathbb{R}$, where $f(t) = \frac{1}{1+t^2}$.

- If $s = 0$, $\hat{f}(s) = \pi$. For $s \neq 0$, choose μ_R^\pm depending on s :
- $|e^{-isz}| = e^{s \operatorname{Im}(z)} : s > 0 \text{ use } \mu_R^-, \quad s < 0 \text{ use } \mu_R^+$.

$$s > 0 : \quad \hat{f}(s) = \lim_{R \rightarrow \infty} \int_{[-R,R]+\mu_R^-} \frac{e^{-isz}}{1+z^2} dz$$
$$= -2\pi i \operatorname{Res}\left(\frac{e^{-isz}}{1+z^2}, -i\right) = \pi e^{-s}$$

$$s < 0 : \quad \hat{f}(s) = \lim_{R \rightarrow \infty} \int_{[-R,R]+\mu_R^+} \frac{e^{-isz}}{1+z^2} dz$$
$$= 2\pi i \operatorname{Res}\left(\frac{e^{-isz}}{1+z^2}, i\right) = \pi e^s$$

Piece these together: $\hat{f}(s) = \pi e^{-|s|}$.

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Piece these together: $\hat{f}(s) = \pi e^{-|s|}$.

Evaluate: $\hat{f}(s)$ for $s \neq 0$, where $f(t) = \frac{1}{1-it}$.

$$\begin{aligned} s > 0 : \quad \hat{f}(s) &= \lim_{R \rightarrow \infty} \int_{[-R,R]+\mu_R^-} \frac{e^{-isz}}{1-iz} dz \\ &= -2\pi i \operatorname{Res}\left(\frac{e^{-isz}}{1-iz}, -i\right) = 2\pi e^{-s} \end{aligned}$$

$$\begin{aligned} s < 0 : \quad \hat{f}(s) &= \lim_{R \rightarrow \infty} \int_{[-R,R]+\mu_R^+} \frac{e^{-isz}}{1-iz} dz \\ &= 0 \quad (\text{no poles}) \end{aligned}$$

For $s = 0$: $\int_{-\infty}^{\infty} \frac{1}{1-it} dt$ depends on how you take limits.

Example: for symmetric limits $\lim_{R \rightarrow \infty} \int_{-R}^R \frac{1}{1-it} dt = \pi$

Evaluate: $\hat{f}(s)$ for $s \neq \pm 1$, where $f(t) = \frac{\sin t}{t}$.

$$\hat{f}(s) = \lim_{R \rightarrow \infty} \int_{[-R,R]} \frac{e^{-i(s-1)z} - e^{-i(s+1)z}}{2iz} dz$$

$$s > 1 : \quad \hat{f}(s) = \lim_{R \rightarrow \infty} \int_{[-R,R] + \mu_R^-} \frac{e^{-i(s-1)z} - e^{-i(s+1)z}}{2iz} dz = 0$$

$$s < -1 : \quad \hat{f}(s) = \lim_{R \rightarrow \infty} \int_{[-R,R] + \mu_R^+} \frac{e^{-i(s-1)z} - e^{-i(s+1)z}}{2iz} dz = 0$$

$-1 < s < 1$: “go around 0” to handle exponentials separately.

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{[-R,R]_-} \frac{e^{-i(s-1)z} - e^{-i(s+1)z}}{2iz} dz &= 2\pi i \operatorname{Res}\left(\frac{e^{-i(s-1)z}}{2iz}, 0\right) \\ &= \pi \end{aligned}$$

Evaluate: $\hat{f}(s)$ for $s \neq \pm 1$, where $f(t) = \frac{\sin t}{t}$.

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