Lecture 22: Conformal Mappings

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Conformal mappings on the plane

Consider a differentiable mapping $f(x,y) = \big(u(x,y), v(x,y)\big)$, and the linearization of f at a point (x,y) :

$$\begin{bmatrix} u(x+\Delta x, y+\Delta y) \\ v(x+\Delta x, y+\Delta y) \end{bmatrix} \approx \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} + Df(x,y) \cdot \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

where $\quad Df(x,y) = \begin{bmatrix} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y) \end{bmatrix}$

Geometric Definition of conformal

The map f is conformal if, at each (x,y), the matrix $Df(x,y)$ is non-singular and angle-preserving. Equivalently,

$$Df(x,y) = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where r and θ depend on (x,y), and $r \neq 0$.

Lemma

$f(x, y) = \big(u(x, y), v(x, y)\big)$ is conformal if and only if the C-R equations, $u_x = v_y$ and $u_y = -v_x$ hold, and $Df \neq 0$.

Proof. If $\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$ then C-R eqn's hold.

Conversely, C-R eqn's say $\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ takes the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

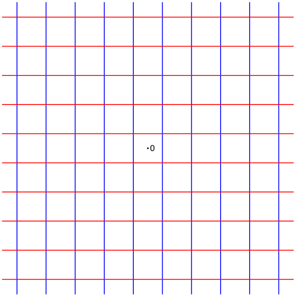
The point (a, b) lies on the circle of radius $r = \sqrt{a^2 + b^2}$, so we can write $a = r\cos\theta$, $b = r\sin\theta$, for some θ. □

Corollary

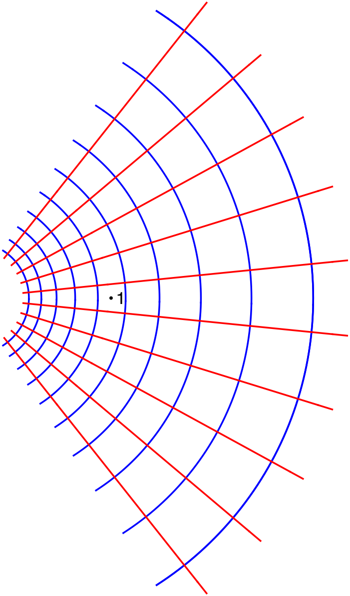
An analytic function $f(z)$ is conformal at points where $f'(z) \neq 0$, where we identify the complex numbers $\mathbb{C}$ with the plane $\mathbb{R}^2$.

Remark. If write $f'(z) = r\,e^{i\theta}$, then get the same r, θ above.

Grid representation of conformal map $z \to e^z$

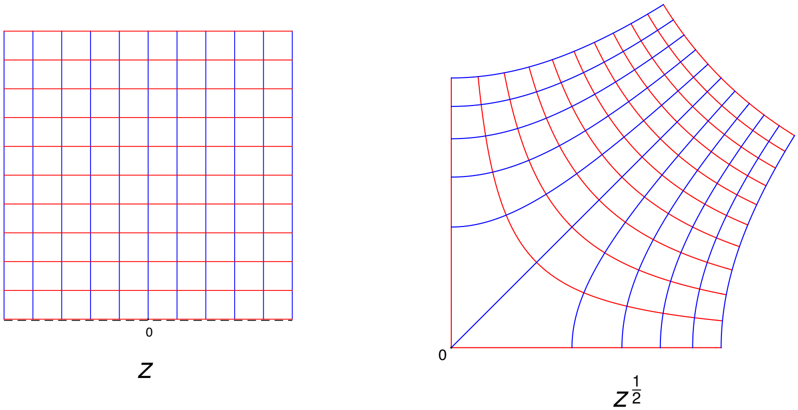


z



e^z

Grid representation of conformal map $z \rightarrow z^{\frac{1}{2}}$



Conformal equivalence

Definition

We say two open sets U and V in $\mathbb{C}$ are conformally equivalent if there is an analytic map $f : U \to V$ that is 1-1 and onto. Such an f is called a conformal equivalence between U and V.

- $f^{-1}(w)$ is then a conformal equivalence between V and U.

Examples.

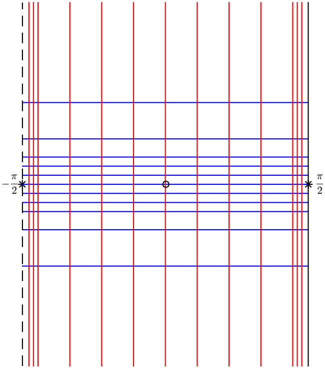
- $f(z) = e^z$ is a conformal equivalence between

$$U = \{z : -\pi < \operatorname{Im}(z) < \pi\} \quad \text{and} \quad V = \mathbb{C} \setminus (-\infty, 0]$$

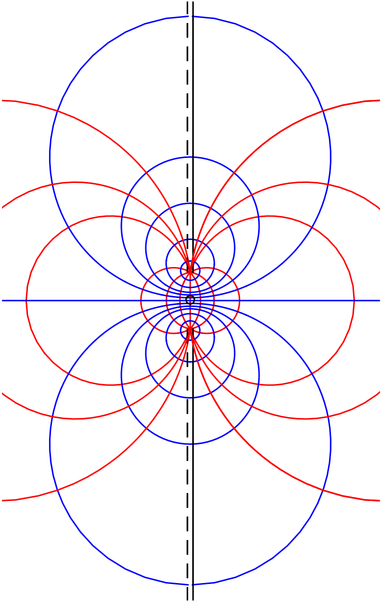
- $f(z) = z^{\frac{1}{2}}$ is a conformal equivalence between

$$U = \{z : \operatorname{Im}(z) > 0\} \ \text{ and } \ V = \{w : \operatorname{Im}(w) > 0 \text{ and } \operatorname{Re}(w) > 0\}$$

$\{z : \mathrm{Re}(z) \in (-\frac{\pi}{2}, \frac{\pi}{2})\} \to \mathbb{C} \setminus (-i\infty, -i] \cup [i, i\infty)$



z



$w = \tan z$