Lecture 23: Conformal Equivalences

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Conformal equivalence

Definition

We say two open sets U and V in $\mathbb{C}$ are conformally equivalent if there is an analytic map $f : U \to V$ that is 1-1 and onto. Such an f is called a conformal equivalence between U and V.

- $f^{-1}(w)$ is then a conformal equivalence between V and U.

Examples.

- $f(z) = e^z$ gives conformal equivalences between

$$U = \{z : -\pi < \text{Im}(z) < \pi\} \quad \text{and} \quad V = \mathbb{C} \setminus (-\infty, 0]$$

$$U = \{z : -\tfrac{\pi}{2} < \text{Im}(z) < \tfrac{\pi}{2}\} \quad \text{and} \quad V = \{z : \text{Re}(z) > 0\}$$

- $f(z) = z^{\frac{1}{2}}$ (principal) is a conformal equivalence between

$$U = \{z : \text{Im}(z) > 0\} \;\; \text{and} \;\; V = \{z : \text{Im}(z) > 0 \text{ and } \text{Re}(z) > 0\}$$

The disc and the right half-plane

$$w = \frac{1+z}{1-z}, \qquad z = \frac{w-1}{w+1}$$

give a conformal equivalence of $z \in \mathbb{C} \setminus \{1\}$ and $w \in \mathbb{C} \setminus \{-1\}$

$$w = \frac{(1+z)(1-\overline{z})}{|1-z|^2} = \frac{1-|z|^2+2i\,\mathrm{Im}(z)}{|1-z|^2}$$

$$\begin{aligned} \mathrm{Re}\big(w\big) &> 0 \quad \text{if} \quad |z| < 1 \\ \mathrm{Re}\big(w\big) &= 0 \quad \text{if} \quad |z| = 1 \\ \mathrm{Re}\big(w\big) &< 0 \quad \text{if} \quad |z| > 1 \end{aligned}$$

Is conformal equivalence of $z \in D_1(0)$ and $\{w : \mathrm{Re}(w) > 0\}$

The disc and the upper half-plane

$$w = i\,\frac{1+z}{1-z}\,, \qquad z = \frac{w-i}{w+i}$$

gives a conformal equivalence of $z \in \mathbb{C} \setminus \{1\}$ and $w \in \mathbb{C} \setminus \{-i\}$

$$\begin{aligned} |z| < 1 \quad &\text{if} \quad |w-i| < |w+i| \\ |z| = 1 \quad &\text{if} \quad |w-i| = |w+i| \\ |z| > 1 \quad &\text{if} \quad |w-i| > |w+i| \end{aligned}$$

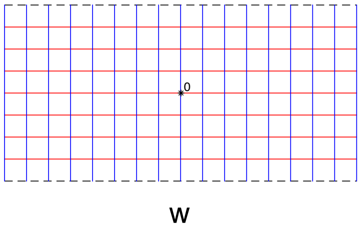
Is a conformal equivalence of $\{w : \mathrm{Im}(w) > 0\}$ and $z \in D_1(0)$

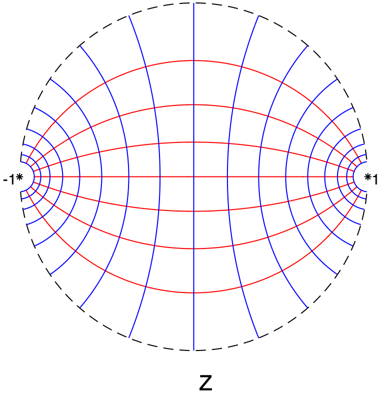
Composition of conformal equivalences

$$z = \frac{e^w - 1}{e^w + 1}, \qquad w = \log\left(\frac{1+z}{1-z}\right)$$

conf. equiv. of $\left\{w : -\frac{\pi}{2} < \mathrm{Im}(w) < \frac{\pi}{2}\right\}$ and $z \in D_1(0)$.

- This takes $w = -\infty$ to $z = -1$, and $w = +\infty$ to $z = 1$.



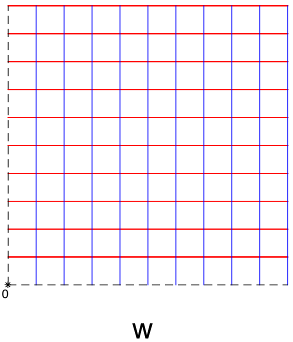


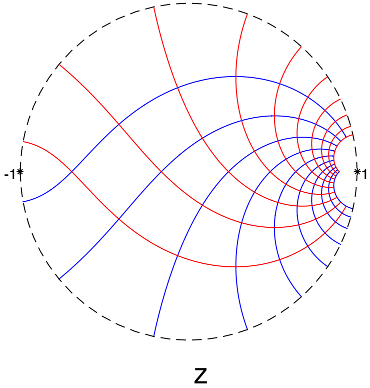
Composition of conformal equivalences

$$z = \frac{w^2 - i}{w^2 + i}, \qquad w = \sqrt{i\,\frac{1+z}{1-z}}$$

conf. equiv. of $\{\mathrm{Re}(w) > 0\} \cap \{\mathrm{Im}(w) > 0\}$ and $z \in D_1(0)$.

- This takes $w = 0$ to $z = -1$, and $w = \infty$ to $z = 1$.





Classification of conformal equivalences of $\mathbb{C}$ and $\mathbb{C}$

Theorem

An entire, 1-1 function on $\mathbb{C}$ is of the form $f(z) = az + b$, for some constants $a, b \in \mathbb{C}$ with $a \neq 0$.

Proof. Write $f(z) = \sum_{k=0}^{\infty} a_k z^k$.

- $f(1/z) = \sum_{k=0}^{\infty} a_k z^{-k}$ is 1-1 on $\mathbb{C} \setminus \{0\}$, and has isolated singularity at 0.
- $f(1/z)$ can't have an essential singularity since it is 1-1, by Theorem 3.4.12 and the Open Mapping Theorem.
- $f(z)$ is thus a polynomial, and must be linear since it is 1-1.