Lecture 24: Linear Fractional Transformations

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Definition

Linear Fractional Transformation (LFT): $f(z) = \dfrac{az+b}{cz+d}$

where $a, b, c, d \in \mathbb{C}$, and $ad - bc \neq 0$.

- $f(z)$ does not change if multiply (a, b, c, d) by same number.
- If $c = 0$, then $f(z)$ is linear. If $c \neq 0$, simple pole at $z = -d/c$.
- We set $f(\infty) = \lim_{z\to\infty} f(z) = a/c$, and $f(\infty) = \infty$ if $c = 0$.

  We define $f(-d/c) = \infty$ (so again $f(\infty) = \infty$ if $c = 0$).

  Then $f(z)$ is now defined as a map $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$.
- Fixed points: for $z \in \mathbb{C}$, $f(z) = z \Leftrightarrow az + b = cz^2 + dz$.

  If f has 3 or more fixed points (including ∞), then $f(z) = z$.

Associate to matrix a LFT: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad f(z) = \dfrac{az+b}{cz+d}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \quad \Rightarrow \quad f\big(g(z)\big) \quad \text{where} \quad g(z) = \frac{a'z+b'}{c'z+d'}$$

In particular: $\quad f^{-1}(z) = \dfrac{dz-b}{-cz+a}$

Theorem

Given two sets of 3 points $\{z_0, z_1, z_2\}$, $\{w_0, w_1, w_2\} \subset \mathbb{C} \cup \{\infty\}$, there exists a unique LFT such that $f(z_j) = w_j$ for $j = 0, 1, 2$.

Uniqueness: f, g two such maps, then $f \circ g^{-1}$ has 3 distinct fixed points, so $f\big(g^{-1}(w)\big) = w$, hence $f(z) = g(z)$.

Conformal automorphisms of $\mathbb{D} = \{z : |z| < 1\}$

If $|b| < 1$, consider the map h_b

$$h_b(z) = \frac{z-b}{1-\bar{b}z} \qquad \text{for which} \qquad h_b^{-1} = h_{-b}.$$

- Pole of h_b is at $z = 1/\bar{b} \in \{z : |z| > 1\}$, so h_b analytic on $\mathbb{D}$.
- If $z \in \partial\mathbb{D}$, so $z\bar{z} = 1$, then

$$|h_b(z)| = \left| \frac{1}{z} \frac{z-b}{\bar{z}-\bar{b}} \right| = 1$$

- By the Maximum Modulus Theorem, $|h_b(z)| < 1$ if $|z| < 1$.
- Same holds for h_b^{-1}, so

Fact

h_b is a 1-1, analytic map of $\mathbb{D}$ onto $\mathbb{D}$, $h_b(b) = 0$, $h_b(0) = -b$.

Conformal automorphisms of $\mathbb{D} = \{z : |z| < 1\}$

Theorem

Every conformal equivalence from $\mathbb{D}$ to $\mathbb{D}$ must be of the form

$$f(z) \;=\; e^{i\theta}\,\frac{z-b}{1-\bar{b}z} \quad \text{for some } \;\theta \in [0, 2\pi)\,,\; b \in \mathbb{D}.$$

Proof. If $f : \mathbb{D} \to \mathbb{D}$ is 1–1, onto, and $f(b) = 0$, let $g = f \circ h_{-b}$.

$$g : \mathbb{D} \xrightarrow{\text{1–1, onto}} \mathbb{D} \quad \text{and} \quad g(0) = 0 \quad \Rightarrow \quad g(z) = e^{i\theta}\, z$$

for some $\theta \in [0, 2\pi)$ by Theorem 3.5.6. Then

$$f(z) \;=\; g\big(h_b(z)\big) \;=\; e^{i\theta}\, h_b(z)\,.$$