# Lecture 25: Circles, Lines, and LFT's 

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## Lines and Circles in $\mathbb{C}$

Recall: if $z=x+i y, \quad w=u+i v$,

$$
\operatorname{Re}(\bar{w} z)=u x+v y=(u, v) \cdot(x, y)
$$

- A line $L$ perpendicular to $w$ is given by:

$$
\operatorname{Re}(\bar{w} z)=c, \quad \text { where } c=\operatorname{Re}\left(\bar{w} z_{0}\right) \text { for any } z_{0} \in L
$$

Circle with center $z_{0}$, radius $r:\left|z-z_{0}\right|^{2}=r^{2}$,

$$
|z|^{2}-2 \operatorname{Re}\left(\bar{z}_{0} z\right)+\left|z_{0}\right|^{2}=r^{2}
$$

The general form of a circle:

$$
|z|^{2}-2 \operatorname{Re}\left(\bar{z}_{0} z\right)=c, \quad \text { where } c=r^{2}-\left|z_{0}\right|^{2}
$$

## Theorem. Suppose $f$ is a linear fractional transformation.

If $E$ is a circle or line in $\mathbb{C}$, then its image $f(E)$ is a circle, unless $E$ passes through the pole of $f$, in which case $f(E)$ is a line.

Proof. First check for $f(z)=1 / z$.
$\mathrm{E}=$ Line: given by $\operatorname{Re}(\bar{w} z)=c$, image is $z: \operatorname{Re}\left(\bar{w} z^{-1}\right)=c$,

$$
\text { Multiply by } z \bar{z}=|z|^{2}: \operatorname{Re}(w z)=c|z|^{2}
$$

Line if $c=0$ (so $E$ passes through origin), circle if $c \neq 0$.
$\mathbf{E}=$ Circle: $|z|^{2}-2 \operatorname{Re}\left(\bar{z}_{0} z\right)=c$, image is $z$ such that

$$
\left|z^{-1}\right|^{2}-2 \operatorname{Re}\left(\bar{z}_{0} z^{-1}\right)=c
$$

Multiply by $z \bar{z}=|z|^{2}$ to turn this into

$$
1-2 \operatorname{Re}\left(z_{0} z\right)=c|z|^{2}
$$

Line if $c=0$ (so $E$ passes through origin), circle if $c \neq 0$.

## General case:

- Each LFT is either linear, or of the form: $f(z)=L_{1}\left(\frac{1}{L_{2}(z)}\right)$ for linear maps $L_{1}, L_{2}$.
- Linear maps take circles and lines to circles and lines.

Easiest way to determine image: use three points
Example: $f(z)=\frac{z+1}{z-1}$

- $E=\left\{z:\left|z-\frac{1}{2}\right|=\frac{1}{2}\right\} \supset\left\{0,1, \frac{1+i}{2}\right\}$ Image contains $\{-1, \infty,-1-2 i\}$, so it's the line $\operatorname{Re}(z)=-1$
- $E=\{z: \operatorname{Re}(z)=0\} \supset\{0, i,-i\}$ Image contains $\{-1,-i, i\}$, so it's the circle $|z|^{2}=1$

