Lecture 26: Conformal automorphisms of $\mathbb{D}$ (and $\mathbb{H}$)

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Previously:

- Every 1-1 conformal map of $\mathbb{C} \to \mathbb{C}$ is of the form

$$f(z) \;=\; az + b\,, \qquad a,\, b \in \mathbb{C}.$$

With the proper interpretation of conformal:

- Every 1-1 conformal map of $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ has form

$$f(z) \;=\; \frac{az+b}{cz+d}\,, \qquad a,\, b,\, c,\, d \in \mathbb{C}.$$

Note: the maps from $\mathbb{C} \to \mathbb{C}$ are those for which $f(\infty) = \infty$.

Conformal automorphisms of $\mathbb{D} = \{z : |z| < 1\}$

If $|b| < 1$, consider the map h_b

$$h_b(z) = \frac{z - b}{1 - \bar{b}z} \qquad \text{for which} \qquad h_b^{-1} = h_{-b}.$$

- Pole of h_b is at $z = 1/\bar{b} \in \{z : |z| > 1\}$, so h_b analytic on $\mathbb{D}$.
- If $z \in \partial\mathbb{D}$, so $z\bar{z} = 1$, then

$$|h_b(z)| = \left| \frac{1}{z} \, \frac{z - b}{\bar{z} - \bar{b}} \right| = 1$$

- By the Maximum Modulus Theorem, $|h_b(z)| < 1$ if $|z| < 1$.
- Same holds for h_b^{-1}, so

Fact

h_b is a 1-1, analytic map of $\mathbb{D}$ onto $\mathbb{D}$, $h_b(b) = 0$, $h_b(0) = -b$.

Conformal automorphisms of $\mathbb{D} = \{z : |z| < 1\}$

Theorem

Every conformal equivalence from $\mathbb{D}$ to $\mathbb{D}$ must be of the form

$$f(z) \;=\; e^{i\theta}\,\frac{z-b}{1-\bar{b}z} \quad \text{for some } \;\theta \in [0, 2\pi)\,,\; b \in \mathbb{D}.$$

Proof. If $f : \mathbb{D} \to \mathbb{D}$ is 1–1, onto, and $f(b) = 0$, let $g = f \circ h_{-b}$.

$$g(0) : \mathbb{D} \xrightarrow{\text{1–1, onto}} \mathbb{D} \quad \text{and} \quad g(0) = 0 \;\;\Rightarrow\;\; g(z) = e^{i\theta}\, z$$

for some $\theta \in [0, 2\pi)$ by Theorem 3.5.6. Then

$$f(z) \;=\; g\big(h_b(z)\big) \;=\; e^{i\theta}h_b(z)\,.$$

Conformal automorphisms of $\mathbb{H} = \{z : \mathrm{Im}(z) > 0\}$

Theorem

The conformal equivalences from $\mathbb{H}$ to $\mathbb{H}$ are of the form

$$f(z) \;=\; \frac{az+b}{cz+d}\,, \qquad a,\, b,\, c,\, d \in \mathbb{R}, \quad ad - bc > 0.$$

Proof Suppose $f : \mathbb{H} \xrightarrow{\text{1–1, onto}} \mathbb{H}$. We will show f is a linear fractional transformation. Necessarily $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, so it must have real values for a, b, c, d.

To see f must be an LFT, recall $g(z) = \dfrac{z-i}{z+i} \;:\; \mathbb{H} \xrightarrow{\text{1–1, onto}} \mathbb{D}$.

$$g \circ f \circ g^{-1} : \mathbb{D} \xrightarrow{\text{1–1, onto}} \mathbb{D} \quad \Rightarrow \quad f = g^{-1} \circ \left(e^{i\theta} h_b\right) \circ g$$

Finally: $\mathrm{Im}\big(f(z)\big) \;=\; \dfrac{(ad-bc)\,\mathrm{Im}(z)}{|cz+d|^2}$, so $ad - bc > 0$.