# Lecture 3: Cauchy’s Theorem for Cycles 

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## Chains and Cycles

- A chain is a finite collection of paths: $\Gamma=\left\{\gamma_{1}, \ldots, \gamma_{n}\right\}$.
- $\Gamma$ and $\Gamma^{\prime}$ are equivalent in $E$ if for all $f$ continuous on $E$

$$
\int_{\Gamma} f(w) d w=\int_{\Gamma^{\prime}} f(w) d w
$$

- We say a chain $\Gamma$ is a cycle if we can turn $\Gamma$ into a collection of closed paths by successively joining together paths.

To verify a chain is a cycle: show the set of its left endpoints (counted with multiplicity) equals the set of its right endpoints.

Example: $\Gamma=\{[0,1],[0,-i],[i, 0],[1, i],[-1,0],[-i,-1]\}$
Left ends: $\{0,0, i, 1,-1,-i\} \quad$ Right ends: $\{1,-i, 0, i, 0,-1\}$

## The index function of a cycle

## Definition

If $\Gamma=\left\{\gamma_{1}, \ldots, \gamma_{n}\right\}$ is a cycle, and $z$ does not lie on $\Gamma$, we define

$$
\operatorname{ind}_{\Gamma}(z)=\frac{1}{2 \pi i} \int_{\Gamma} \frac{1}{w-z} d w=\sum_{j=1}^{n} \frac{1}{2 \pi i} \int_{\gamma_{j}} \frac{1}{w-z} d w
$$

- We do not require that each $\gamma_{j}$ be closed. However...
- $\Gamma$ is equivalent to a collection of closed paths $\left\{\tilde{\gamma}_{1}, \ldots, \tilde{\gamma}_{m}\right\}$,

$$
\operatorname{ind}_{\Gamma}(z)=\sum_{j=1}^{m} \operatorname{ind}_{\tilde{\gamma}_{j}}(z) \quad \text { is an integer. }
$$

- $\operatorname{ind}_{\Gamma}(z)$ is constant on each component of $\mathbb{C} \backslash\{\Gamma\}$.


## Most important example

Recall: $\partial D_{r}\left(z_{0}\right)=\gamma$, where $\gamma(t)=z_{0}+r e^{i t}, t \in[0,2 \pi]$.
Example: boundary of the annulus $r<\left|z-z_{0}\right|<R$ :

$$
\Gamma=\left\{\partial D_{R}\left(z_{0}\right),-\partial D_{r}\left(z_{0}\right)\right\}
$$

$$
\operatorname{ind}_{\Gamma}(z)= \begin{cases}0, & \left|z-z_{0}\right|<r, \\ 1, & r<\left|z-z_{0}\right|<R, \\ 0, & \left|z-z_{0}\right|>R .\end{cases}
$$

## Cauchy Theorem for general sets

Suppose $E \subset \mathbb{C}$ is open, $f$ is analytic on $E$, and $\Gamma$ is a cycle in $E$ such that $\operatorname{ind}_{\Gamma}(z)=0$ for all $z \notin E$. Then $\int_{\Gamma} f(w) d w=0$.

- Taking $f(w)=(w-z)^{-1}$ with $z \notin E$ shows $\operatorname{ind}_{\Gamma}(z)=0$ for $z \notin E$ is necessary for the result to hold.
- If $\Gamma$ is equivalent to $\left\{\tilde{\gamma}_{1}, \ldots, \tilde{\gamma}_{m}\right\}$, where each $\tilde{\gamma}_{j} \subset E_{j}$, with $E_{j}$ a convex open subset of $E$, the theorem holds since

$$
\int_{\Gamma} f(w) d w=\sum_{j=1}^{m} \int_{\tilde{z}_{j}} f(w) d w
$$

and each term $=0$ by Cauchy Theorem for convex sets.

- The condition $\operatorname{ind}_{\Gamma}(z)=0$ for all $z \notin E$ implies that $\Gamma$ is equivalent to such a collection $\left\{\tilde{\gamma}_{1}, \ldots, \tilde{\gamma}_{m}\right\}$. We will verify the equivalence for all cases we are interested in.

Example 1. $E=\mathbb{C} \backslash\{0\}$, and $\Gamma=\left\{\partial D_{R}(0),-\partial D_{r}(0)\right\}$.

- Split each circle into 4 pieces (one piece for each quadrant).
- Add the line segments $[r, R],[-r,-R]$, $[i r, i R]$, $[-i r,-i R]$, and also their negatives, to the collection; the result can be written as 4 closed paths, one in each quadrant.

Example 2. $E=\left\{z: r_{0}<|z|<R_{0}\right\}$,

$$
\Gamma=\left\{\partial D_{R}(0),-\partial D_{r}(0)\right\}, \quad r_{0}<r<R<R_{0} .
$$

- Same idea as above, but may need to split into many wedges.

