

Lecture 4: Cauchy Integral Formula for Cycles

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Cauchy's Theorem for general sets

Suppose $E \subset \mathbb{C}$ is open, f is analytic on E , and Γ is a cycle in E such that $\text{ind}_\Gamma(z) = 0$ for all $z \notin E$. Then $\int_\Gamma f(w) dw = 0$.

Examples.

- Annulus: center $z = 2$, inner radius $\frac{1}{2}$, outer radius 4:

$$E = \left\{ z : \frac{1}{2} < |z - 2| < 4 \right\},$$

$$\Gamma = \partial D_{3.5}(2) - \partial D_1(2).$$

- Punctured plane, removed $z = -1$ and $z = 4$,

$$E = \mathbb{C} \setminus \{-1, 4\},$$

$$\Gamma = \partial D_{10}(0) - \partial D_1(-1) - \partial D_1(4).$$

Cauchy Integral Formula for general sets

Suppose $E \subset \mathbb{C}$ is open, f is analytic on E , and Γ is a cycle in E such that $\text{ind}_\Gamma(z) = 0$ for all $z \notin E$. Then

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z} dw = \begin{cases} \text{ind}_\Gamma(z) \cdot f(z), & z \in E \setminus \{\Gamma\}, \\ 0, & z \notin E. \end{cases}$$

Proof. If $z \notin E$, follows directly by Cauchy's theorem.

For $z \in E \setminus \{\Gamma\}$: apply Cauchy's Theorem to $\frac{f(w) - f(z)}{w - z}$

Remark. $\frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z} dw = 0$

whenever z is in the unbounded component of $E \setminus \Gamma$.

For the following: let $g(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w-z} dw$ for $z \notin \Gamma$.

- $f(w) = \frac{1}{1+w^2}$, analytic on $E = \mathbb{C} \setminus \{-i, i\}$,

$$\Gamma = \partial D_2(0) - \partial D_{\frac{1}{2}}(i) - \partial D_{\frac{1}{2}}(-i)$$

$$g(z) = \begin{cases} (1+z^2)^{-1}, & z \in D_2(0) \setminus (\overline{D_{\frac{1}{2}}(i)} \cup \overline{D_{\frac{1}{2}}(-i)}) \\ 0, & z \in D_{\frac{1}{2}}(i) \cup D_{\frac{1}{2}}(-i) \cup \{z : |z| > 2\} \end{cases}$$

- $f(w) = \log w$, analytic on $E = \mathbb{C} \setminus (-\infty, 0]$, $\Gamma = \partial D_1(2)$

$$g(z) = \begin{cases} \log z, & z \in D_1(2) \\ 0, & z \in \mathbb{C} \setminus \overline{D_1(2)} \end{cases}$$

- $f(w) = \tan(w)$, on $E = \mathbb{C} \setminus \{\pi(n + \frac{1}{2}) : n \in \mathbb{Z}\}$

$$\Gamma = \partial D_4(0) - \partial D_2(0)$$

$$g(z) = \begin{cases} \tan(z), & \text{if } 2 < |z| < 4 \\ 0, & \text{if } |z| < 2 \text{ or if } |z| > 4 \end{cases}$$

Gives integral formula for $\tan(w)$ on the annulus $2 < |z| < 4$:

$$\tan(z) = \frac{1}{2\pi i} \int_{|w|=4} \frac{\tan(w)}{w-z} dw - \frac{1}{2\pi i} \int_{|w|=2} \frac{\tan(w)}{w-z} dw$$

Both terms are analytic functions of z ; together give $\tan(z)$.

Theorem

Suppose that f is a continuous function on $\{z : |z - z_0| = r\}$.

Then $g(z) = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(w)}{w - z} dw$ is analytic on $\mathbb{C} \setminus \partial D_r(z_0)$, and admits a convergent expansion on each component:

$$\text{For } |z - z_0| < r : \quad g(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

$$\text{For } |z - z_0| > r : \quad g(z) = - \sum_{k=-\infty}^{-1} a_k (z - z_0)^k$$

$$\text{where in both cases: } a_k = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(w)}{(w - z_0)^{k+1}} dw$$

Proof. Proof for $|z - z_0| < r$ is exactly like 427 Lecture 21:

$$\frac{1}{w - z} = \frac{1}{w - z_0} \frac{1}{1 - \left(\frac{z - z_0}{w - z_0}\right)} = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(w - z_0)^{k+1}}$$

converges uniformly on the set $|w - z_0| = r$ if $|z - z_0| < r$.

Proof for $|z - z_0| > r$. Suppose $|w - z_0| = r$, $|z - z_0| > r$:

$$\frac{1}{w - z} = \frac{-1}{z - w} = \frac{-1}{z - z_0} \frac{1}{1 - \left(\frac{w - z_0}{z - z_0}\right)} = -\sum_{j=0}^{\infty} \frac{(w - z_0)^j}{(z - z_0)^{j+1}}$$

converges uniformly on the set $|w - z_0| = r$ if $|z - z_0| < r$. So

$$\int_{\partial D_r(z_0)} \frac{f(w)}{w - z} dw = -\sum_{j=0}^{\infty} \left(\int_{\partial D_r(z_0)} f(w) (w - z_0)^j dw \right) \frac{1}{(z - z_0)^{j+1}}$$

Set $j + 1 = -k$: $\frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(w)}{w - z} dw = -\sum_{k=-\infty}^{-1} a_k (z - z_0)^k$,

where:

$$a_k = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(w)}{(w - z_0)^{k+1}} dw$$