Lecture 5: Laurent Expansions on Annuli

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From Lecture 4

Theorem

Suppose that f is a continuous function on $\{z : |z - z_0| = r\}$.

Then $g(z) = \dfrac{1}{2\pi i} \displaystyle\int_{\partial D_r(z_0)} \frac{f(w)}{w - z}\, dw$ is analytic on $\mathbb{C} \setminus \partial D_r(z_0)$,

and admits a convergent expansion on each component:

For $|z - z_0| < r$: $\quad g(z) = \displaystyle\sum_{k=0}^{\infty} a_k (z - z_0)^k$

For $|z - z_0| > r$: $\quad g(z) = -\displaystyle\sum_{k=-\infty}^{-1} a_k (z - z_0)^k$

where in both cases: $\quad a_k = \dfrac{1}{2\pi i} \displaystyle\int_{\partial D_r(z_0)} \frac{f(w)}{(w - z_0)^{k+1}}\, dw$

Bounds on $|a_k|$ and convergence of series

Suppose that $|f(w)| \leq M$ on $\partial D_r(z_0)$. Then:

$$|a_k| \leq \frac{1}{2\pi} \sup_{|w-z_0|=r} \left(\frac{|f(w)|}{|w-z_0|^{k+1}} \right) \ell(\partial D_r(z_0)) = M\, r^{-k}.$$

- $\displaystyle\sum_{k=0}^{\infty} a_k (z-z_0)^k$ converges absolutely for $|z-z_0| < r$

  and converges uniformly on $\{z : |z-z_0| \leq s\}$ if $s < r$.

- $\displaystyle\sum_{k=-\infty}^{-1} a_k (z-z_0)^k$ converges absolutely for $|z-z_0| > r$

  and converges uniformly on $\{z : |z-z_0| \geq s\}$ if $s > r$.

Theorem: we allow $r_0 = 0$ and/or $R_0 = \infty$, and any $r \in (r_0, R_0)$

If f is analytic on $\{z : r_0 < |z - z_0| < R_0\}$, then on this set

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \qquad a_k = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(w)}{(w - z_0)^{k+1}}\, dw$$

- $\displaystyle\sum_{k=0}^{\infty} a_k (z - z_0)^k$ converges for $|z - z_0| < R_0$,
- $\displaystyle\sum_{k=-\infty}^{-1} a_k (z - z_0)^k$ converges for $|z - z_0| > r_0$.

Remark. $a_k = \dfrac{1}{2\pi i} \displaystyle\int_{\partial D_r(z_0)} \frac{f(w)}{w - z_0}\, dw$ is same for all $r \in (r_0, R_0)$

by Cauchy's Theorem applied to $\Gamma = \partial D_r(z_0) - \partial D_{r'}(z_0)$.

Proof: consider any r, R with $r_0 < r < R < R_0$

For z with $r < |z - z_0| < R$: by Cauchy Integral Formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial D_R(z_0)} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(w)}{w - z}\, dw$$

1st term $= \sum_{k=0}^{\infty} a_k (z - z_0)^k$, which converges for $|z - z_0| < R$;

this is true for any $R < R_0$, so series converges on $|z - z_0| < R_0$

2nd term $= \sum_{k=-\infty}^{-1} a_k (z - z_0)^k$, which converges for $|z - z_0| > r$;

this is true for any $r > r_0$, so series converges on $|z - z_0| > r_0$

Example

$$f(z) = \frac{1}{(z-1)(z-2i)} \quad \text{on the annulus} \quad 1 < |z| < 2.$$

Partial fractions: $f(z) = \dfrac{1}{1-2i}\left(\dfrac{1}{z-1} - \dfrac{1}{z-2i}\right)$

- $|z| > 1$: $\dfrac{1}{z-1} = \dfrac{1}{z}\dfrac{1}{1-z^{-1}} = \dfrac{1}{z}\displaystyle\sum_{j=0}^{\infty}(z^{-1})^j = \sum_{k=-\infty}^{-1} z^k$

- $|z| < 2$: $\dfrac{1}{z-2i} = \dfrac{\frac{i}{2}}{\frac{i}{2}z+1} = \frac{i}{2}\displaystyle\sum_{k=0}^{\infty}(-1)^k\left(\tfrac{i}{2}z\right)^k$

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k, \qquad a_k = \begin{cases} \frac{1}{1-2i}, & k \le -1 \\ \frac{1}{1-2i}(-1)^k\left(\frac{i}{2}\right)^{k+1}, & k \ge 0 \end{cases}$$

Example

$$f(z) = \frac{1}{(z-1)(z-2i)} \quad \text{on the annulus} \quad 2 < |z| < \infty.$$

Partial fractions: $f(z) = \dfrac{1}{1-2i}\left(\dfrac{1}{z-1} - \dfrac{1}{z-2i}\right)$

- $|z| > 1$: $\dfrac{1}{z-1} = \dfrac{1}{z}\dfrac{1}{1-z^{-1}} = \dfrac{1}{z}\displaystyle\sum_{j=0}^{\infty}(z^{-1})^j = \sum_{k=-\infty}^{-1} z^k$

- $|z| > 2$: $\dfrac{1}{z-2i} = \dfrac{1}{z}\dfrac{1}{1-2iz^{-1}} = \displaystyle\sum_{j=0}^{\infty}\frac{(2i)^j}{z^{j+1}} = \sum_{k=-\infty}^{-1}\frac{z^k}{(2i)^{k+1}}$

$$f(z) = \sum_{k=-\infty}^{-1} a_k z^k, \qquad a_k = \frac{1}{1-2i}\left(1 + \frac{1}{(2i)^{k+1}}\right)$$

Example

$$f(z) = \frac{1}{(z-1)(z-2i)} \quad \text{on the disk} \quad |z| < 1.$$

Partial fractions: $f(z) = \dfrac{1}{1-2i}\left(\dfrac{1}{z-1} - \dfrac{1}{z-2i}\right)$

- $|z| < 1$: $\dfrac{1}{z-1} = \dfrac{-1}{1-z} = (-1)\displaystyle\sum_{k=0}^{\infty} z^k$

- $|z| < 2$: $\dfrac{1}{z-2i} = \dfrac{\frac{i}{2}}{\frac{i}{2}z+1} = \frac{i}{2}\displaystyle\sum_{k=0}^{\infty}(-1)^k\left(\tfrac{i}{2}z\right)^k$

$$f(z) = \sum_{k=0}^{\infty} a_k\, z^k\,, \qquad a_k = \frac{1}{1-2i}\left(-1 + (-1)^k\left(\tfrac{i}{2}\right)^{k+1}\right)$$