

Lecture 6: Laurent expansions about a point

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Math 428, Winter 2020

Most important case $r_0 = 0$: isolated singularity at z_0 .

Theorem

If f is analytic on $D_R(z_0) \setminus \{z_0\}$, then $f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$,

- $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ converges on $D_R(z_0)$ and is analytic there,
- $\sum_{k=-\infty}^{-1} a_k(z - z_0)^k$ converges on $\mathbb{C} \setminus \{z_0\}$ and is analytic there.

The **principal part** is $\sum_{k=-\infty}^{-1} a_k(z - z_0)^k$

- Removable singularity: principal part = 0.
- Pole order m : $a_{-m} \neq 0$, $a_k = 0$ if $k < -m$,
- Essential singularity: $a_k \neq 0$ infinitely many k .

- $f(z) = \sin\left(\frac{1}{z}\right)$: analytic on $0 < |z| < \infty$.

$$\sin\left(\frac{1}{z}\right) = \sum_{j=0}^{\infty} \frac{1}{(2j+1)!} \left(\frac{1}{z}\right)^{2j+1} \quad a_k = \begin{cases} \frac{1}{(-k)!}, & k \leq -1, \text{ odd} \\ 0, & \text{otherwise} \end{cases}$$

- $f(z) = \frac{1}{z^3 - z}$: analytic on $0 < |z| < 1$.

$$\frac{1}{z^3 - z} = \frac{1}{z} \frac{1}{(z^2 - 1)} = \frac{(-1)}{z} \sum_{j=0}^{\infty} z^{2j} = -\frac{1}{z} - z - z^3 - \dots$$

- Using the Partial Fraction Decomposition:

$$\frac{1}{z^3 - z} = \frac{(-1)}{z} + \frac{z}{z^2 - 1} = -\frac{1}{z} - z \sum_{j=0}^{\infty} z^{2j} = -\frac{1}{z} - z - z^3 - \dots$$

- $\frac{e^z}{(z-1)^2}$: analytic on $0 < |z-1| < \infty$.

When of the form $\frac{f(z)}{(z-z_0)^m}$: expand $f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j$.

Trick: $e^z = e^{(z-1)+1} = e^1 e^{z-1} = e \sum_{j=0}^{\infty} \frac{1}{j!} (z-1)^j$

$$\frac{e^z}{(z-1)^2} = \frac{e}{(z-1)^2} + \frac{e}{(z-1)} + \frac{e}{2!} + \frac{e}{3!}(z-1) + \frac{e}{4!}(z-1)^2 + \dots$$

- $\frac{z^3}{(z-2)^2}$: analytic on $0 < |z-2| < \infty$.

$$\frac{z^3}{(z-2)^2} = \frac{(2+(z-2))^3}{(z-2)^2} = \frac{2^3}{(z-2)^2} + \frac{3 \cdot 2^2}{(z-2)} + 3 \cdot 2 + (z-2)$$

More complicated examples: denominator not a power

- $\frac{1}{e^z - 1}$: analytic on $0 < |z| < 2\pi$.

Let $g(z) = \begin{cases} \frac{e^z - 1}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}$, so $g(0) \neq 0$.

$\frac{1}{e^z - 1} = \frac{1}{z} \frac{1}{g(z)}$, need to Taylor expand $\frac{1}{g(z)}$ about $z = 0$:

$$\frac{1}{g(z)} = \frac{1}{g(0)} - \frac{g'(0)}{g(0)^2} z + \frac{1}{2} \left(\frac{2g'(0)^2}{g(0)^3} - \frac{g''(0)}{g(0)^2} \right) z^2 + \dots$$

$$g(z) = 1 + \frac{1}{2}z + \frac{1}{6}z^2 + \dots \Rightarrow g(0) = 1, g'(0) = \frac{1}{2}, g''(0) = \frac{1}{3}.$$

$$\frac{1}{e^z - 1} = \frac{1}{z} \frac{1}{g(z)} = \frac{1}{z} - \frac{1}{2} + \frac{1}{2} \left(\frac{1}{2} - \frac{1}{3} \right) z + \dots$$

More complicated examples: denominator not a power

- $\frac{1}{e^z - 1}$: analytic on $0 < |z| < 2\pi$.

Manipulation of power series to find first few terms:

$$e^z - 1 = \sum_{k=1}^{\infty} \frac{1}{k!} z^k = z \left(1 + \frac{1}{2}z + \frac{1}{6}z^2 + \dots \right)$$

$$\frac{1}{e^z - 1} = \frac{1}{z} \cdot \frac{1}{1 + (\frac{1}{2}z + \frac{1}{6}z^2 + \dots)} = \frac{1}{z} \sum_{j=0}^{\infty} \left(-\frac{1}{2}z - \frac{1}{6}z^2 - \dots \right)^j$$

- For terms up to $a_1 z$: need only $j \leq 2$, ignore \dots terms,

$$\frac{1}{z} - \frac{1}{2} + \left(\frac{1}{2^2} - \frac{1}{6} \right) z$$