Lecture 7: The Residue at isolated singularities

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Definition

Suppose that f is analytic on $D_R(z_0) \setminus \{z_0\}$ some $R > 0$; that is, f has an isolated singularity at z_0. If $f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k$ is the Laurent expansion of f, we define $\operatorname{Res}(f, z_0) = c_{-1}$.

Reason for interest in c_{-1} : if $0 < r < R$

$$\int_{\partial D_r(z_0)} f(w)\, dw = 2\pi i \operatorname{Res}(f, z_0)$$

Proof.

$$\int_{\partial D_r(z_0)} f(w)\, dw = \sum_{k=-\infty}^{\infty} c_k \int_{\partial D_r(z_0)} (z - z_0)^k\, dw = 2\pi i\, c_{-1}$$

Calculating residues for simple poles

If $f(z)$ has a **simple pole** (i.e. pole order 1) at $z = z_0$:

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Proof. Taking the Laurent expansion on $D_R(z_0) \setminus \{z_0\}$

$$(z - z_0) f(z) \;=\; c_{-1} + \sum_{k=0}^{\infty} c_k (z - z_0)^{k+1}$$

has limit c_{-1} as $z \to z_0$.

$$\operatorname{Res}\left(\frac{1}{\sin z}, k\pi\right) = \lim_{z \to k\pi} \frac{z - k\pi}{\sin z} = \lim_{z \to k\pi} \frac{1}{\cos z} = \frac{1}{\cos(k\pi)} = (-1)^k$$

More examples with simple poles

$$\operatorname{Res}\left(\tan z, \tfrac{\pi}{2}\right) = \lim_{z\to\frac{\pi}{2}} \tan z = \lim_{z\to\frac{\pi}{2}} \frac{(z-\frac{\pi}{2})\sin z}{\cos z}$$

$$= \lim_{z\to\frac{\pi}{2}} \frac{\sin z + (z-\frac{\pi}{2})\cos z}{-\sin z} = \frac{1}{-1} = -1$$

- $\tan z$ is periodic with period $\pi \Rightarrow \operatorname{Res}\left(\tan z, (k+\frac{1}{2})\pi\right) = -1.$

$$\operatorname{Res}\left(\frac{1}{e^z-1}, 2\pi k i\right) = \lim_{z\to 2\pi k i} \frac{z-2\pi k i}{e^z-1} = \lim_{z\to 2\pi k i} \frac{1}{e^z} = \frac{1}{e^{2\pi k i}} = 1$$

Yet another example of simple poles

$$\operatorname{Res}\left(\frac{z+3}{(z+1)(z-2)}, 2\right) = \lim_{z\to 2}(z-2)\frac{z+3}{(z+1)(z-2)} = \frac{5}{3}$$

$$\operatorname{Res}\left(\frac{z+3}{(z+1)(z-2)}, -1\right) = \lim_{z\to -1}(z+1)\frac{z+3}{(z+1)(z-2)} = -\frac{2}{3}$$

Remark. If f has an isolated singularity at z_0, and g is analytic on a neighborhood of z_0, then: $\operatorname{Res}(f+g, z_0) = \operatorname{Res}(f, z_0)$

- Using partial fractions leads to same result as above:

$$\frac{z+3}{(z+1)(z-2)} = \frac{5}{3}\frac{1}{(z-2)} - \frac{2}{3}\frac{1}{(z+1)}$$

For simple poles, taking limits is easier than partial fractions:

$$\operatorname{Res}\left(\frac{z^3+10}{(z^2+1)^3(z-5)}, 5\right) = \lim_{z\to 5} \frac{z^3+10}{(z^2+1)^3} = \frac{5^3+10}{(5^2+1)^3}$$

For residue at $z = i$, can't use this since i is not a simple pole.

- We must factor $(z^2+1)^3 = (z-i)^3(z+i)^3$, so that

$$\frac{z^3+10}{(z^2+1)^3(z-5)} = \frac{g(z)}{(z-i)^3}, \quad g(z) = \frac{z^3+10}{(z+i)^3(z-5)}$$

- We expand $g(z) = g(i) + g'(i)(z-i) + \frac{1}{2}g''(i)(z-i)^2 + \cdots$

$$\operatorname{Res}\left(\frac{z^3+10}{(z^2+1)^3(z-5)}, i\right) = \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{z^3+10}{(z+i)^3(z-5)}\right)\Bigg|_{z=i}$$

Theorem

Suppose f analytic on $D_r(z_0) \setminus \{z_0\}$. If $\mathrm{Res}(f, z_0) = 0$, then one can write $f(z) = F'(z)$ where F is analytic on $D_r(z_0) \setminus \{z_0\}$.

Proof. Since $c_{-1} = 0$, the Laurent expansion of $f(z)$ at z_0 is:

$$f(z) = \sum_{k=-\infty}^{-2} c_k (z - z_0)^k + \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

Define:

$$F(z) = \sum_{k=-\infty}^{-2} \frac{c_k}{k+1} (z - z_0)^{k+1} + \sum_{k=0}^{\infty} \frac{c_k}{k+1} (z - z_0)^{k+1}$$

Then $\displaystyle\int_{z_1}^{z_2} f(w)\, dw = F(z_2) - F(z_1)$ when $z_1, z_2 \in D_R(z_0) \setminus \{z_0\}$ and $z_0 \notin [z_1, z_2]$, so $F(z)$ is analytic and $F'(z) = f(z)$.