# Lecture 9: Counting Zeroes 

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The logarithmic derivative of $f(z)$ is $\frac{f^{\prime}(z)}{f(z)}$

- This is analytic where $f(z)$ is analytic and $f(z) \neq 0$.
- Formally this equals $\log (f(z))^{\prime}$
- Additive rule: $\frac{(h(z) g(z))^{\prime}}{h(z) g(z)}=\frac{h^{\prime}(z)}{h(z)}+\frac{g^{\prime}(z)}{g(z)}$
- Important example: $h(z)=\left(z-z_{0}\right)^{m}, \quad \frac{h^{\prime}(z)}{h(z)}=\frac{m}{z-z_{0}}$

If $f(z)=\left(z-z_{0}\right)^{m} g(z)$, then $\frac{f^{\prime}(z)}{f(z)}=\frac{m}{z-z_{0}}+\frac{g^{\prime}(z)}{g(z)}$

## Theorem

If $f$ has a zero of order $m$ at $z_{0}$, then $\frac{f^{\prime}}{f}$ has a simple pole at $z_{0}$,
and

$$
\operatorname{Res}\left(\frac{f^{\prime}}{f}, z_{0}\right)=m
$$

Proof. Write $f(z)=\left(z-z_{0}\right)^{m} g(z)$, where $g\left(z_{0}\right) \neq 0$. Then

$$
\frac{f^{\prime}(z)}{f(z)}=\frac{m}{z-z_{0}}+\frac{g^{\prime}(z)}{g(z)}
$$

$\frac{g^{\prime}(z)}{g(z)}$ is analytic at $z_{0} \Rightarrow$ pole is simple and $\operatorname{Res}\left(\frac{f^{\prime}}{f}, z_{0}\right)=m$.
Remark. Zeroes of $f$ are isolated $\Rightarrow$ so are singularities of $\frac{f^{\prime}}{f}$

## Examples

- $\sin z$ has zero of order 1 at $z=k \pi, \frac{\sin ^{\prime} z}{\sin z}=\cot z$,

Principal part of $\cot z$ at $k \pi$ is $\frac{1}{z-k \pi}, \quad \operatorname{Res}(\cot z, k \pi)=1$.

- $\cos z$ has zero of order 1 at $z=\left(k+\frac{1}{2}\right) \pi, \frac{\cos ^{\prime} z}{\cos z}=-\tan z$, Principal part of $\tan z$ is $\frac{-1}{z-\left(k+\frac{1}{2}\right) \pi}, \quad \operatorname{Res}(\tan z, k \pi)=-1$.
- Polynomial $P(z)=\left(z-z_{1}\right)^{m_{1}}\left(z-z_{2}\right)^{m_{2}} \cdots\left(z-z_{n}\right)^{m_{n}}$

$$
\frac{P^{\prime}(z)}{P(z)}=\frac{m_{1}}{z-z_{1}}+\frac{m_{2}}{z-z_{2}}+\cdots+\frac{m_{n}}{z-z_{n}}
$$

## Counting Zeroes Theorem

Suppose $f$ is analytic on $E$, and $\Gamma$ a cycle in $E$ with ind ${ }_{\Gamma}(z)=0$ for $z \notin E$ and $f \neq 0$ on $\Gamma$. If $\left\{z_{j}\right\}_{j=1}^{n}$ are the zeroes of $f$ inside the set where $\operatorname{ind}_{\Gamma}(z) \neq 0$, and $m_{j}$ is the multiplicity of $z_{j}$, then

$$
\int_{\Gamma} \frac{f^{\prime}(z)}{f(z)} d z=2 \pi i \sum_{j=1}^{n} \operatorname{ind}_{\Gamma}\left(z_{j}\right) \cdot m_{j}
$$

Proof. Follows by the Residue Theorem: the poles of $\frac{f^{\prime}(z)}{f(z)}$ are exactly the zeroes of $f$; none are contained on $\Gamma$ by assumption.

Special case. If $\gamma$ is a closed path that equals $\partial E$ traversed in the counter-clockwise direction, and $f \neq 0$ on $\partial E$, then

$$
\frac{1}{2 \pi i} \int_{\gamma} \frac{f^{\prime}(z)}{f(z)} d z=\#\{\text { zeroes of } f \text { inside } E\}
$$

where the zeroes are counted with multiplicity.

- $\int_{|z|=1} \frac{2 z}{z^{2}-c} d z=2 \pi i \cdot 2$ if $|c|<1$
- $\int_{|z|=1} \frac{n z^{n-1}}{z^{n}-c} d z=2 \pi i \cdot n \quad$ if $|c|<1$
- If $P(z)$ is a polynomial of order $n$, and $R$ is large enough so that all the zeroes of $P(z)$ are contained inside $|z|<R$

$$
\int_{|z|=R} \frac{P^{\prime}(z)}{P(z)} d z=2 \pi i n
$$

- $\int_{|z|=13} \frac{e^{z}}{e^{z}-1} d z=10 \pi i$

