Math 428, Winter 2020, Homework 1 Solutions

Section 3.5: 7

Solution. There are two ways to do this. First, since $f(0) = f'(0) = 0$, we can write $f(z) = z^2 g(z)$ for $g(z)$ analytic on $D_1(0)$.

By the Schwartz Lemma applied to f, we have $|f(z)| \le |z|$. This tells us $|zg(z)| \le 1$. Since $zg(z)$ vanishes at $z = 0$, we can apply the Schwartz Lemma to $zg(z)$, and deduce $|zg(z)| \le |z|$, so $|g(z)| \le 1$ for all $z \in D_1(0)$. Then $|f(z)| = |z|^2 |g(z)| \le |z|^2$.

Alternatively, repeat the proof of the Schwartz Lemma to show $|g(z)| \le 1$ from the maximum principle. That is, note that $|f(z)| \le 1$ implies $|z|^2|g(z)| \le 1$, so $|g(z)| \le |z|^{-2}$. Taking any $r < 1$, then $|g(z)| \le r^{-2}$ if $|z| = r$, hence by the maximum principle $|g(z)| \le r^{-2}$, provided $|z| \le r < 1$. Given a point $z \in D_1(0)$, you can take the limit as $r \to 1$ to deduce $|g(z)| \le 1$.

Section 3.5: 11

Solution. To solve $\frac{2z-1}{z-2} = w$, write

$$\frac{2z-1}{z-2} = \frac{2(z-2)+3}{z-2} = 2 + \frac{3}{z-2} = w$$

Then

$$z = 2 + \frac{3}{w-2} = \frac{2w-1}{w-2}$$

That is, $f(f(z)) = z$. So if we show that f maps $D_1(0)$ to $D_1(0)$, it follows that f is 1-1 (since if $f(z_1) = f(z_2)$ then $f(f(z_1)) = f(f(z_2))$ so $z_1 = z_2$. It is also onto, since for $|w| < 1$, $z = f(w)$ satisfies $|z| < 1$ and $f(z) = w$.

As noted in the hint, consider $|z| = 1$ and write $1 = z\overline{z}$ to factor

$$\left|\frac{2z-1}{z-2}\right| = \left|\frac{1}{z}\right| \left|\frac{2z-1}{1-2\overline{z}}\right| = \frac{|2z-1|}{|2\overline{z}-1|} = 1$$

f is not constant so by the maximum principle $|f(z)| < 1$ if $|z| < 1$.

Finally, to show $f(0) = \frac{1}{2}$ just plug in $z = 0$.

Section 4.1: 2

Solution. You could draw a picture, or note that the collection of left endpoints equals the collection of right endpoints. (In defining left versus right, they flip for $-\gamma_j$, and count twice for $2\gamma_7$.) That is,

$$\{-1,1,0,0,-1+i,1+i,i,i\}=\{-1+i,1+i,-1,1,i,i,0,0\}$$

Section 4.1: 3

Solution. One example is the single closed path gotten by joining (in order)

$$(\gamma_1,\gamma_5,-\gamma_7,\gamma_4,\gamma_2,-\gamma_6,-\gamma_7,-\gamma_3)$$

Another is to write Γ as the union of the two closed paths

$$(\gamma_1,\gamma_5,-\gamma_7,-\gamma_3)\ +\ (\gamma_4,\gamma_2,-\gamma_6,-\gamma_7)$$

Section 4.1: 13

Solution. The cycle $\partial D_{3.5}(0)-\partial D_{1.5}(0)$ works.

Additional Problems: 1

Solution. Picture is 3 circles of appropriate radius/center; outermost circle counterclockwise, inner 2 circles clockwise.

$$\mathrm{ind}_\Gamma(z)=\begin{cases}1\,, & z\in D_3(0)\setminus\big(\overline{D}_{\frac12}(-1)\cup\overline{D}_{\frac12}(1)\big)\\ 0\,, & z\in D_{\frac12}(-1)\cup D_{\frac12}(1)\cup\big(\mathbb{C}\setminus\overline{D}_3(0)\big)\end{cases}$$

In particular, $\mathrm{ind}_\Gamma(z)=0$ at $z=\pm1$.

Additional Problems: 2

Solution. The contour Γ of problem 2 satisfies $\mathrm{ind}_\Gamma(z)=0$ for all $z\notin E$ (since $z\notin E$ implies $z=\pm1$). So by the Cauchy Theorem

$$0=\int_\Gamma f(z)\ =\int_{\partial D_3(0)}f(w)\,dw-\int_{\partial D_{\frac12}(-1)}f(w)\,dw-\int_{\partial D_{\frac12}(1)}f(w)\,dw$$

whence the result follows.