Math 428, Winter 2020, Homework 2 Solutions

Section 4.2: 2

Solution. $\dfrac{1}{(z+2)(z-3)}$ is analytic on $E = \mathbb{C} \setminus \{-2, 3\}$. Γ satisfies $\text{ind}_\Gamma(z) = 0$ for all $z \notin E$. So $\displaystyle\int_\Gamma \dfrac{1}{(z+2)(z-3)}\, dz = 0$.

Section 4.2: 3

Solution. Part. frac's: $f(z) = \dfrac{1}{(z+2)(z-3)} = \dfrac{1}{5}\left(\dfrac{1}{z-3} - \dfrac{1}{z+2}\right)$. We use the fact that $\displaystyle\int_\gamma \dfrac{1}{z-z_0}\, dz = 2\pi i\, \text{ind}_\gamma(z_0)$ to find

$$\int_{\gamma_1} f(z)\, dz = 0\,, \qquad \int_{\gamma_2} f(z)\, dz = -\dfrac{2\pi i}{5}\,, \qquad \int_{\gamma_3} f(z)\, dz = \dfrac{2\pi i}{5}$$

Section 4.2: 6

Solution. We use the Cauchy Integral Formula, and the fact that

$$\text{ind}_{\gamma_1}\left(\tfrac{1}{2}\right) = 1\,, \text{ind}_{\gamma_1}\left(\tfrac{3}{2}\right) = \text{ind}_{\gamma_1}(3) = 0\,, \quad \text{ind}_{\gamma_2}\left(\tfrac{1}{2}\right) = \text{ind}_{\gamma_2}\left(\tfrac{3}{2}\right) = 1\,, \text{ind}_{\gamma_2}(3) = 0$$

to find

$$\int_\Gamma \dfrac{e^z}{z-\frac{1}{2}}\, dz = 0\,, \qquad \int_\Gamma \dfrac{e^z}{z-\frac{3}{2}}\, dz = 2\pi i e^{\frac{3}{2}}\,, \qquad \int_\Gamma \dfrac{e^z}{z-3}\, dz = 0\,.$$

Section 4.3: 4

Solution. For $z \neq 0$,

$$z^{-3} e^z = \dfrac{1}{z^3} \sum_{j=0}^\infty \dfrac{1}{j!}\, z^j = \sum_{j=0}^\infty \dfrac{1}{j!}\, z^{j-3} = \sum_{k=-3}^\infty \dfrac{1}{(k+3)!}\, z^k$$

Section 4.3: 5

Solution. We use the expansion for $|z| < 1$: $\log(1+z) = \displaystyle\sum_{j=1}^\infty \dfrac{(-1)^{j-1}}{j}\, z^j$, which comes from $\log(1+z)' = (1+z)^{-1} = \displaystyle\sum_{j=0}^\infty (-1)^j\, z^j$, and $\log(1) = 0$. Then

$$\dfrac{\log(1+z)}{z^4} = \sum_{j=1}^\infty \dfrac{(-1)^{j-1}}{j}\, z^{j-4} = \sum_{k=-3}^\infty \dfrac{(-1)^{k-1}}{k+4}\, z^k$$

Section 4.3: 6

Solution. One way (one can also start with partial fractions):

$$\frac{z}{z^2+1} = \frac{1}{z-i}\frac{z}{z+i} = \frac{1}{z-i}\left(1 - \frac{i}{z+i}\right) = \frac{1}{z-i}\left(1 - \frac{i}{2i+(z-i)}\right)$$

$$= \frac{1}{z-i}\left(1 - \frac{\frac{1}{2}}{1 - \frac{i}{2}(z-i)}\right) = \frac{1}{z-i}\left(1 - \frac{1}{2}\sum_{j=0}^{\infty}(\tfrac{i}{2})^j(z-i)^j\right) = \frac{\frac{1}{2}}{z-i} - \sum_{k=0}^{\infty}\frac{i^{k+1}(z-i)^k}{2^{k+2}}$$

Additional Problem: 1

Solution. The series converges uniformly on the image of γ, so we can interchange summation and integration to write

$$\int_\gamma f(z)\,dz = \sum_{k=-\infty}^{\infty} a_k \int_\gamma z^k\,dz$$

If $k \geq 0$, then z^k is analytic (on all of $\mathbb{C}$), so $\int_\gamma z^k\,dz = 0$ by Cauchy's theorem.

If $k \leq -2$, then we can write $z^k = \left(\frac{z^{k+1}}{k+1}\right)'$, so by the fundamental theorem of calculus and the fact that γ is closed we have $\int_\gamma z^k\,dz = 0$.

Thus, the only term that is not zero is $k = -1$. So we get

$$\int_\gamma f(z)\,dz = a_{-1}\int_\gamma \frac{1}{z}\,dz = a_{-1}\,2\pi i\,\mathrm{ind}_\gamma(0).$$

An alternative proof can be made by splitting the sum into two terms to write $f(z) = a_{-1}z^{-1} + g(z)$, where

$$g(z) = \sum_{k=-\infty}^{-2} a_k z^k + \sum_{k=0}^{\infty} a_k z^k.$$

By the last result from Lecture 7, we can write $g(z) = G'(z)$ for $G(z)$ analytic on the annulus $\{1 < |z| < 2\}$. Then $\int_\gamma g(z)\,dz = 0$ by the fundamental theorem of calculus, so we again get

$$\int_\gamma f(z)\,dz = a_{-1}\int_\gamma z^{-1}\,dz = a_{-1}\,2\pi i\,\mathrm{ind}_\gamma(0).$$