Math 428, Winter 2020, Homework 3 Solutions

Section 4.4: 1

Solution. $f(z) = \dfrac{1}{2(z-\frac{1}{2})(z-2)}$. Simple poles at $z = \frac{1}{2}$, $z = 2$.

$$\mathrm{Res}(f,2) = \lim_{z\to 2} \frac{z-2}{2(z-\frac{1}{2})(z-2)} = \frac{1}{3}, \qquad \mathrm{Res}(f,\tfrac{1}{2}) = \lim_{z\to \frac{1}{2}} \frac{z-\frac{1}{2}}{2(z-\frac{1}{2})(z-2)} = -\frac{1}{3}.$$

Section 4.4: 3

Solution. $f(z) = \dfrac{e^z}{z^2-1} = \dfrac{e^z}{(z-1)(z+1)}$, $\mathrm{Res}(f,1) = \dfrac{e}{2}$, $\mathrm{Res}(f,-1) = -\dfrac{e^{-1}}{2}$,

$$\int_{|z|=2} \frac{e^z}{z^2-1}\, dz \;=\; 2\pi i\Big(\frac{e}{2} - \frac{e^{-1}}{2}\Big)$$

Section 4.4: 8

Solution. Contour $|z-5| = 4$ encloses (simple) poles of $f(z) = \dfrac{\log z}{\sin z}$ at $z = \pi$ and $z = 2\pi$.

$$\mathrm{Res}(f,\pi) = \lim_{z\to\pi} \frac{(z-\pi)\log z}{\sin z} = \frac{\log \pi}{\cos \pi} = -\log \pi\,.$$

$$\mathrm{Res}(f,2\pi) = \lim_{z\to 2\pi} \frac{(z-2\pi)\log z}{\sin z} = \frac{\log(2\pi)}{\cos(2\pi)} = \log(2\pi)\,.$$

$$\int_{|z-5|=4} \frac{\log z}{\sin z}\, dz = 2\pi i\big(\log(2\pi) - \log \pi\big) = 2\pi i \log 2.$$

Section 4.5: 1

Solution. On the set $|z| = 1$, we have $|3z^7| = 3$, and $|-z^3+1| \le |z^3| + 1 \le 2$. So we apply Rouché's Theorem with $g(z) = 3\,z^7$, and $f(z) = 3z^7 - z^3 + 1$ to see that $f(z)$ has 7 zeroes inside the unit disc, since $g(z)$ does.

Section 4.5: 2

Solution. On the set $|z| = 1$, we have $|-4z^3| = 4$, and $|z^5 + z - 1| \leq |z^5| + |z| + 1 \leq 3$. So we apply Rouché's Theorem with $g(z) = -4z^3$, and $f(z) = z^5 - 4z^3 + z - 1$ to see that $f(z)$ has 3 zeroes inside the unit disc, since $g(z)$ does.

Section 4.5: 4

Solution. We are given $h(z)$ is nonzero on the boundary of the disc, which means that the quantity $c = \min_{|z|=1} |h(z)|$ satisfies $c > 0$. Let $M = \max_{|z|=1} |g(z)|$. then, if $M|\lambda| < c$ we have $|\lambda g(z)| \leq |h(z)|$ for all $z \in \partial D_1(0)$. By Rouche's theorem it follows that h and $h + \lambda g$ have the same number of zeroes inside $|z| < 1$. Now note that the condition $M|\lambda| < c$ is equivalent to $|\lambda| < \delta$, where $\delta = c/M$.

Additional problem 1.

Solution. For this, we use the following fact from the proof of Rouché's Theorem:

$$\#\big\{\text{zeroes of } f \text{ inside } \gamma\big\} \;-\; \#\big\{\text{zeroes of } g \text{ inside } \gamma\big\} \;=\; \mathrm{ind}_{h\circ\gamma}(0)$$

where $h(z) = \dfrac{f(z)}{g(z)}$. The condition $\mathrm{Re}\Big(\dfrac{f(z)}{g(z)}\Big) > 0$ for $z \in \{\gamma\}$ says that the path $h \circ \gamma$ is contained in the half-space $\mathrm{Re}(z) > 0$, which is a convex set that does not contain the point 0. Thus $\mathrm{ind}_{h\circ\gamma}(0) = 0$.

Additional problem 2.

Solution. $f(z) = \dfrac{1}{1+z^2} = \dfrac{1}{(z-i)(z+i)} = \dfrac{-\frac{1}{2}i}{z-i} + \dfrac{\frac{1}{2}i}{z+i}$ has poles at $z = \pm i$, and $\mathrm{Res}(f, i) = -\frac{1}{2}i$, $\mathrm{Res}(f, -i) = \frac{1}{2}i$.

Suppose that γ_0 and γ_1 are two paths in $\mathbb{C} \setminus [-i, i]$ that both start at z_0 and end at z_1. Consider the closed path $\gamma = \gamma_1 - \gamma_0$. Since $\{\gamma\} \subset \mathbb{C} \setminus [-i, i]$ then f has no poles on $\{\gamma\}$, so

$$\begin{aligned}\int_\gamma \frac{1}{1+z^2}\, dz \;&=\; 2\pi i\Big(-\frac{i}{2}\mathrm{ind}_\gamma(i) + \frac{i}{2}\mathrm{ind}_\gamma(-i)\Big) \\ &=\; \pi\Big(\mathrm{ind}_\gamma(i) - \mathrm{ind}_\gamma(-i)\Big)\end{aligned}$$

Since $[-i, i] \subset \mathbb{C} \setminus \{\gamma\}$, it follows that we can connect $-i$ to i by a path that does not intersect γ. Hence $\mathrm{ind}_\gamma(i) = \mathrm{ind}_\gamma(-i)$. We conclude that $\int_\gamma f(z)\, dz = 0$ for all closed paths γ contained in $\mathbb{C} \setminus [-i, i]$. Thus,

$$\int_{\gamma_1 - \gamma_0} \frac{1}{1+z^2}\, dz = \int_{\gamma_1} \frac{1}{1+z^2}\, dz \;-\; \int_{\gamma_0} \frac{1}{1+z^2}\, dz \;=\; 0.$$