

Math 428, Winter 2020, Homework 5 Solutions

Section 5.2: 4

Solution.

$$\int_0^\infty \frac{x^2}{(1+x^2)^2} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2}{(1+x^2)^2} dx = \pi i \operatorname{Res}\left(\frac{z^2}{(1+z^2)^2}, i\right)$$

$$\operatorname{Res}\left(\frac{z^2}{(z-i)^2(z+i)^2}, i\right) = \left.\left(\frac{z^2}{(z+i)^2}\right)'\right|_{z=i} = \frac{2i}{(2i)^2} - \frac{2 \cdot i^2}{(2i)^3} = \frac{1}{4i}$$

$$\int_0^\infty \frac{x^2}{(1+x^2)^2} dx = \frac{\pi}{4}$$

Section 5.2: 5

Solution. $z^2+2z+2 = (z+1)^2+1$, roots are $z_1 = -1+i$, $z_2 = -1-i$.

$$\int_{-\infty}^\infty \frac{x}{(x^2+2x+2)^2} dx = 2\pi i \operatorname{Res}\left(\frac{z}{(z^2+2z+2)^2}, z_1\right)$$

$$\operatorname{Res}\left(\frac{z}{(z-z_1)^2(z-z_2)^2}, z_1\right) = \left.\left(\frac{z}{(z-z_2)^2}\right)'\right|_{z=z_1}$$

$$= \frac{1}{(z_1-z_2)^2} - \frac{2z_1}{(z_1-z_2)^3}$$

$$= \frac{1}{(2i)^2} - \frac{-2+2i}{(2i)^3}$$

$$= \frac{2}{(2i)^3} = \frac{i}{4}$$

$$\int_{-\infty}^\infty \frac{x}{(x^2+2x+2)^2} dx = -\frac{\pi}{2}$$

Section 5.2: 9

Solution. We need to complete the contour to the upper half plane $\operatorname{Im}(z) > 0$ and get

$$\int_{-\infty}^\infty \frac{e^{it}}{(1+t^2)^2} dt = 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{(1+z^2)^2}, i\right)$$

There is one pole in $\operatorname{Im}(z) > 0$, at $z = i$ and we calculate

$$\operatorname{Res}\left(\frac{e^{iz}}{(z+i)^2(z-i)^2}, i\right) = \left.\frac{d}{dz} \frac{e^{iz}}{(z+i)^2}\right|_{z=i} = \pi e^{-1}.$$

Then $\int_{-\infty}^\infty \frac{\cos t}{(1+t^2)^2} dt$ is the real part, which is πe^{-1} .

Section 5.3: 2

Solution. I use the definition without the factor $1/\sqrt{2\pi}$.

If $s > 0$, then since $\operatorname{Re} -isz = s \operatorname{Im}(z)$ is negative if $\operatorname{Im}(z) < 0$, we complete the contour in $\operatorname{Im}(z) < 0$ (which will be clockwise hence index -1 at the pole) and

$$\int_{-\infty}^{\infty} \frac{ze^{-isz}}{1+z^2} dz = -2\pi i \operatorname{Res}\left(\frac{ze^{-isz}}{1+z^2}, -i\right) = -\pi i e^{-s}, \quad s > 0.$$

If $s < 0$ we complete the contour in $\operatorname{Im}(z) > 0$, and

$$\int_{-\infty}^{\infty} \frac{ze^{-isz}}{1+z^2} dz = 2\pi i \operatorname{Res}\left(\frac{ze^{-isz}}{1+z^2}, i\right) = \pi i e^s, \quad s < 0.$$

Section 5.3: 5

Solution. $z^2 + 4z + 5 = (z+2)^2 + 1$ has simple roots $z_1 = -2 + i$ in the upper half space, and $z_2 = -2 - i$ in the lower half space. For $s \geq 0$ we proceed as above and write

$$\hat{f}(s) = \int_{[-\infty, \infty]} \frac{e^{-isz}}{z^2 + 4z + 5} dz = -2\pi i \operatorname{Res}\left(\frac{e^{-isz}}{z^2 + 4z + 5}, z_2\right)$$

Since $z_2 - z_1 = -2i$,

$$\hat{f}(s) = -2\pi i \frac{e^{-s+2is}}{-2i} = \pi e^{-s} e^{2is}.$$

For $s \leq 0$, we can use $\hat{f}(s) = \overline{\hat{f}(-s)}$ to write, for all s ,

$$\hat{f}(s) = \pi e^{-|s|} e^{2is}.$$

Section 6.1: 4

Solution. The map $w = i \frac{1+z}{1-z}$ is a conformal equivalence of the disc $\{|z| < 1\}$ to the upper half-plane $\{\operatorname{Im}(w) > 0\}$, and the map $w^{\frac{1}{4}}$ (principal branch) is a conformal equivalence of $\{\operatorname{Im}(w) > 0\}$ to the wedge $0 < \arg(z) < \pi/4$. So the desired map is

$$f(z) = \left(i \frac{1+z}{1-z}\right)^{\frac{1}{4}}.$$

Section 6.1: 5

Solution. By example 6.1.4, the map $w = \frac{1+z}{1-z}$ maps the unit disc 1-1 onto the set $\operatorname{Re}(w) > 0$. We write

$$\operatorname{Im}\left(\frac{1+z}{1-z}\right) = \operatorname{Im}\left(\frac{(1+z)(1-\bar{z})}{|1-z|^2}\right) = \frac{2\operatorname{Im}(z)}{|1-z|^2}$$

So the upper disc $\operatorname{Im}(z) > 0$ goes to the first quadrant $\operatorname{Im}(w) > 0$, and the lower half $\operatorname{Im}(z) < 0$ to the 4th quadrant $\operatorname{Im}(w) < 0$.