Math 428 Midterm Solutions, Winter 2019

1. Find all poles of the following functions, find their order, and find the residue of f at each of the poles.

(a.) $f(z) = \dfrac{z}{e^z - 1}$. **Solution**. Isolated singularities at zeroes of $e^z - 1$, so at $z = 2\pi ik$, which are all simple zeroes. $k = 0$ is removable, all other k are simple poles. By l'Hopitâl's rule, $\mathrm{Res}(f, 2\pi ik) = 2\pi ik$.

(b.) $f(z) = \dfrac{z^2}{(z^2-4)^2}$. **Solution**. Isolated singularities at $z = \pm 2$, both give poles of order 2. To find $\mathrm{Res}(f,2)$, write $f(z) = g(z)/(z-2)^2$, where $g(z) = z^2/(z+2)^2$. Then $\mathrm{Res}(f,2) = g'(2) = 1/8$. Similarly, $\mathrm{Res}(f,-2) = g'(-2) = -1/8$.

2. Find the Laurent expansion of $\dfrac{\sin z - z}{z^6}$ about $z = 0$, and use it to find $\mathrm{Res}\Big(\dfrac{\sin z - z}{z^6}, 0\Big)$.
Solution. $\sin z - z = \sum_{k=1}^{\infty} (-1)^k z^{2k+1}/(2k+1)!$, so the Laurent expansion at 0 is

$$\frac{\sin z - z}{z^6} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k-5}$$

The z^{-1} term comes from $k = 2$, so $\mathrm{Res}(f, 0) = 1/5! = 1/120$.

3. Calculate the following contour integrals. In part (b.) $\log z$ is the principal branch.

(a.) $\displaystyle\int_{|z-i|=1} \frac{e^z}{z^2+1}\,dz$. **Solution**. Simple pole at i is only pole inside the curve. $\mathrm{Res}\big(e^z/(z^2+1), i\big) = e^i/2i$, so integral equals $\pi e^i = \pi\big(\cos(1) + i\sin(1)\big)$.

(b.) $\displaystyle\int_{|z-1|=\frac{1}{2}} \frac{e^z}{\log z}\,dz$. **Solution**. Simple pole at 1 is the only pole in the domain. And by l'Hopitâl's rule, $\mathrm{Res}\big(e^z/\log z, 1\big) = e$, so integral equals $2\pi ie$.

4. (a.) Find the maximum modulus of $f(z) = e^z$ on the set $|z| \le 2$. **Solution**. $|e^z| = e^{\mathrm{Re}(z)}$, which is increasing in $\mathrm{Re}(z)$, so we need to find the point in $\{|z| \le 2\}$ (which in fact must be on the boundary) having the largest value of $\mathrm{Re}(z)$. This is $z = 2$. Then $|e^z| = e^2$ there.

(b.) Show that if c is a complex number with $|c| < 3$ then $z^5 - ce^z$ has fives zeroes in $|z| < 2$. **Solution**. If $|z| = 2$, then $|ce^z| = |c|e^{\mathrm{Re}(z)} \le 3 \cdot e^2 \le 27$. But $|z^5| = |z|^5 = 32$ when $|z| = 2$. So, $|(z^5 - ce^z) - z^5| < |z^5|$ on $|z| = 2$. This implies $z^5 - ce^z \ne 0$ when $|z| = 2$. So we can apply Rouché's theorem to see that $z^5 - ce^z$ has the same number of zeroes as z^5 inside $|z| < 2$, which is 5.