

## Math 524, Autumn 2008, Homework 1

The following homework is due Wednesday, October 1.

1. For an integer  $q \geq 2$ , let  $\mathbb{Z}_q = \{0, 1, \dots, q-1\}$ , and let  $\mathbb{Z}_q^{\mathbb{N}}$  denote the collection of sequences  $a = a_1 a_2 a_3 \dots$  with  $a_j \in \mathbb{Z}_q$ . Give  $\mathbb{Z}_q^{\mathbb{N}}$  the *lexicographic* ordering:

$$a > b \iff \text{there exists } k \geq 1 \text{ such that } \begin{cases} a_n = b_n, & n < k \\ a_k > b_k \end{cases}$$

For  $a \in \mathbb{Z}_q^{\mathbb{N}}$ , define

$$Ta = \sum_{n=1}^{\infty} \frac{a_n}{q^n}.$$

Show that  $T$  maps  $\mathbb{Z}_q^{\mathbb{N}}$  onto the closed interval  $[0, 1]$ , that  $Ta \geq Tb$  if  $a \geq b$ , and that if  $a > b$  then

$$Ta = Tb \iff \text{there exists } k \geq 1 \text{ such that } \begin{cases} a_n = b_n, & n < k \\ a_k = b_k + 1 \\ a_n = 0, b_n = q - 1, & n > k \end{cases}$$

(For  $q = 10$ , this says that every real number  $x \in [0, 1]$  admits an infinite decimal expansion  $x = .a_1 a_2 a_3 \dots$ , which is unique except for equalities like  $.2500\bar{0} = .2499\bar{9}$ .)

For the following three problems you may use the fundamental axioms for the real numbers, as well as the Bolzano-Weirstrass property. You may also use elementary algebra and elementary properties of continuous functions (use the  $\epsilon$ - $\delta$  definition of continuity), such as continuity  $\Leftrightarrow$  sequential continuity, but don't use other results about compactness.

2. Establish the *intermediate value theorem*: if  $f(x)$  is a continuous function on  $[a, b]$ , and  $f(a) < 0 < f(b)$ , then there exists  $x \in [a, b]$  such that  $f(x) = 0$ .

3. Show that a continuous function on a closed interval  $[a, b]$  must be bounded; i.e. there exists  $M$  such that  $f(x) \leq M$  for all  $x \in [a, b]$ .

4. For a bounded real-valued function  $f(x)$  defined on an interval  $(a, b)$ , define the upper and lower envelope functions by the rule:

$$\bar{f}(x) = \inf_{\epsilon > 0} \sup_{|y-x| < \epsilon} f(y), \quad \underline{f}(x) = \sup_{\epsilon > 0} \inf_{|y-x| < \epsilon} f(y),$$

and define the *oscillation function* for  $f$  as

$$\text{osc}f(x) = \bar{f}(x) - \underline{f}(x).$$

Show that

- a. The function  $f$  is continuous at  $x$  if and only if  $\text{osc}f(x) = 0$ .
- b. For each  $\epsilon > 0$ , the set  $\{x : \text{osc}f(x) < \epsilon\}$  is an open set.