Math 524, Autumn 2007, Homework 2

The following homework is due Wednesday, October 10.

1. As in the last homework set, let $\mathbb{Z}_q^{\mathbb{N}}$ denote the collection of sequences $a = a_1 a_2 a_3 \ldots$ with $a_j \in \mathbb{Z}_q$. Define

$$\rho(a,b) = \sum_{j=1}^{\infty} \frac{|a_j - b_j|}{q^j}\,.$$

(a.) Show that ρ is a metric on $\mathbb{Z}_q^{\mathbb{N}}$.

(b.) Show that $(\mathbb{Z}_q^{\mathbb{N}}, \rho)$ is a complete metric space. (Hint: start by showing that a sequence of elements $\{a^n\} \subset \mathbb{Z}_q^{\mathbb{N}}$ is Cauchy if and only if the elements "stabilize"; that is, for each j then a_j^n is constant for n large.)

(c.) Show that $(\mathbb{Z}_q^{\mathbb{N}}, \rho)$ is totally bounded.

(d.) For any $a \in \mathbb{Z}_q^{\mathbb{N}}$, show that the set $\{b \in \mathbb{Z}_q^N : b < a\}$ is an open set, and that if a is a finite sequence (i.e. $a_j = 0$ for j large) then it is also a closed set.

2. Let $E \subset \mathbb{Z}_3^{\mathbb{N}}$ denote the subset of sequences so that for each j either $a_j = 0$ or $a_j = 2$.

(a.) Show that E is a closed subset of $(\mathbb{Z}_3^{\mathbb{N}}, \rho)$.

(b.) Show that $Ta = \sum_{j=1}^{\infty} \frac{a_j}{3^j}$ is a one-to-one map from E to a closed subset of $[0,1]$. Indeed, show that if $a, b \in E$ then $\frac{1}{3}\rho(a,b) \le |Ta - Tb| \le \rho(a,b)$.

(c.) Show that the image $T(E)$ is the **Cantor set**, which is obtained by removing the middle third $(\frac{1}{3}, \frac{2}{3})$ from $[0,1]$, then removing the middle thirds $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ from the remaining two intervals, and so on.

3. Let (X,ρ) and (Y,d) be metric spaces, and assume (X,ρ) is compact. Suppose that $f : X \to Y$ is a continuous map of X **onto** Y.

(a.) Show that f is a *quotient map*, in that U is open in Y iff $f^{-1}(U)$ is open in X.

(b.) If f is one-to-one, show that both f and f^{-1} are uniformly continuous.

4. Suppose that $f_n(x)$ is a sequence of non-negative real valued continuous functions on a compact metric space which decreases monotonically pointwise to 0. That is, we assume $f_1(x) \ge f_2(x) \ge f_3(x) \ldots \ge 0$, and for each x we have $\lim_{n\to\infty} f_n(x) = 0$.

Show $f_n \to 0$ uniformly; that is, $\forall \epsilon > 0$ there exists N such that $f_n(x) < \epsilon \ \ \forall x$ if $n \ge N$.

5. Show that a metric space (X,ρ) is complete if and only if the following holds: whenever A_j is a sequence of closed, nonempty subsets of X such that $A_1 \supseteq A_2 \supseteq A_3 \cdots$, and such that $\mathrm{diam}(A_j) \to 0$, then

$$\cap_{j=1}^{\infty} A_j = \{x\}\,, \qquad \text{some } x \in X\,.$$

6. For a function $f : X \to \mathbb{R} \cup \{-\infty\}$ on a metric space X, we define

$$\limsup_{y\to x} f(y) = \inf_{\epsilon>0} \sup_{y \in B(\epsilon,x)} f(y)\,.$$

We say that f is *upper-semicontinuous* if for each $x \in X$ we have

$$\limsup_{y \to x} f(y) = f(x)\,.$$

(a.) Show that f is upper-semicontinuous iff for each x the following holds: given $\epsilon > 0$ there exists $\delta > 0$ so that $f(y) < f(x) + \epsilon$ if $\rho(y, x) < \delta$.

(b.) Show that the function f is upper-semicontinuous iff for each real number λ the set $\{y : f(y) < \lambda\}$ is open in X.

(c.) Show that if f and g are upper-semicontinuous, so are $\max(f, g)$ and $f + g$.

(d.) Show that, if $f_n(x)$ is a sequence of upper-semicontinuous functions, then the function $f(x) = \inf_n f_n(x)$ is also upper-semicontinuous.

(e.) Show that the conclusion of problem 4 holds if the condition of continuity is replaced by upper-semicontinuity.

7. Let (X, ρ) be a metric space, and $A \subseteq X$. Show that a set $U \subseteq A$ is open in the metric space $(A, \rho|_A)$ iff there exists an open set $\tilde{U} \subseteq X$ such that $\tilde{U} \cap A = U$.