Math 524, Autumn 2007, Homework 3

The following homework is due Wednesday, October 17.

1. For $n \geq 1$ define the projections $\Pi_n : \mathbb{Z}_q^{\mathbb{N}} \to \mathbb{Z}_q^n$ by the rule

$$\Pi_n(a_1 a_2 a_3 \ldots) = (a_1, a_2, a_3, \ldots, a_n)$$

(a.) Show that the collection $\mathcal{A}$ of sets of the form $\Pi_n^{-1}(E)$, for $n \in \mathbb{N}$ and $E \subseteq \mathbb{Z}_q^n$, form an algebra of sets.

(b.) Show that each set in $\mathcal{A}$ is both open and closed in $\mathbb{Z}_q^{\mathbb{N}}$ (in the metric topology of last week's homework.)

(c.) Show that the σ-algebra generated by $\mathcal{A}$ is the Borel σ-algebra on $\mathbb{Z}_q^{\mathbb{N}}$.

2. Define a function m on $\mathcal{A}$ by the rule

$$m(\Pi_n^{-1}(E)) = \frac{1}{q^n}\text{card}(E)$$

when $E \in \mathbb{Z}_q^n$ and card(E) is the number of elements in E. Verify that m is well defined on $\mathcal{A}$, and is finitely additive.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. Show that the set of points $x \in R$ such that f is continuous at x is a G_δ set. (See page 22 for the definition of G_δ. This problem is a two-line proof based on problem 4 of the first homework.)

4. Let f be an upper-semicontinuous function on a metric space. For a real number α, what can you say about the sets $\{x : f(x) > \alpha\}$, $\{x : f(x) \geq \alpha\}$, $\{x : f(x) < \alpha\}$, $\{x : f(x) \leq \alpha\}$, and $\{x : f(x) = \alpha\}$? Express your answer as open, closed, F_σ, G_δ, $F_{\sigma\delta}$, or $G_{\delta\sigma}$.

5. Suppose that $f_n(x)$ is a sequence of continuous real-valued functions on a metric space, such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all x. Show that, for real valued α, the set $\{x : f(x) < \alpha\}$ is F_σ. (Hint: write $\lim_{n\to\infty} f(x) = \limsup_{n\to\infty} f(x)$.)

6. Let $\{f_n(x)\}$ be a sequence of continuous real valued functions on a metric space. Show that the set of points x such that $\lim_{n\to\infty} f_n(x)$ exists is a $F_{\sigma\delta}$ set. (Equivalently, the complement of this set is $G_{\delta\sigma}$.)

7. Problem 9, page 27 of Folland.

8. Problem 10, page 27 of Folland.