Math 524, Autumn 2007, Homework 9

The following homework is due Friday, December 7.

1. A Borel measure μ on $\mathbb{R}^n$ is called *regular* if $\mu(K) < \infty$ for all compact sets K, and if for all Borel sets E we have

$$\mu(E) = \sup\{\mu(K) \,:\, K \subseteq E\,, \text{K compact}\} = \inf\{\mu(U) \,:\, U \supseteq E\,, \text{U open}\}$$

Show that if μ and ν are regular Borel measures, and

$$\int \phi \, d\mu = \int \phi \, d\nu$$

for all $\phi \in C_c(\mathbb{R}^n)$, then $\mu = \nu$. [Consider functions $\phi = 1$ on K and $\phi = 0$ on U^c.]

2. If $f \in L^1(\mathbb{R}, dx)$, show that

$$\int_{x_1 < x_2 < \cdots < x_n} f(x_1) f(x_2) \cdots f(x_n) \, dx_1 \, dx_2 \, \cdots dx_n = \frac{1}{n!} \left(\int f(x) \, dx \right)^n$$

[Hint: consider how the integral behaves under permutation of the x_i's.]

3. Let $f(x)$ be a non-negative Lebesgue measurable function on $\mathbb{R}$, and let

$$\phi(t) = m\{x : f(x) > t\}$$

Show that ϕ is right-continuous and decreasing, and that

$$\int_0^\infty \phi(t) \, dt = \int f(x) \, dx$$

4.

(a.) If $f\,, g \in L^1(\mathbb{R}, dx)$, show that $f(x-y)g(y) \in L^1(\mathbb{R}, dx)$ for almost all x.

(b.) If $h(x) = \int f(x-y)g(y) \, dy$ (where defined), show that $h \in L^1(\mathbb{R}, dx)$ and

$$\int |h| \, dx \le \left(\int |f| \, dx \right) \left(\int |g| \, dx \right)$$