Axioms for the Real Numbers

Field Axioms: there exist notions of addition and multiplication, and additive and multiplicative identities and inverses, so that:

- **(P1)** (Associative law for addition): $a + (b + c) = (a + b) + c$
- **(P2)** (Existence of additive identity): $\exists\, 0 :\ a + 0 = 0 + a = a$
- **(P3)** (Existence of additive inverse): $a + (-a) = (-a) + a = 0$
- **(P4)** (Commutative law for addition): $a + b = b + a$
- **(P5)** (Associative law for multiplication): $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- **(P6)** (Existence of multiplicative identity): $\exists\, 1 \neq 0 :\ a \cdot 1 = 1 \cdot a = a$
- **(P7)** (Existence of multiplicative inverse): $a \cdot a^{-1} = a^{-1} \cdot a = 1 \quad \text{for} \quad a \neq 0$
- **(P8)** (Commutative law for multiplication): $a \cdot b = b \cdot a$
- **(P9)** (Distributive law): $a \cdot (b + c) = a \cdot b + a \cdot c$

Order Axioms: there exists a subset of positive numbers P such that

- **(P10)** (Trichotomy): exclusively either $a \in P$ or $-a \in P$ or $a = 0$.
- **(P11)** (Closure under addition): $a, b \in P \Rightarrow a + b \in P$
- **(P12)** (Closure under multiplication): $a, b \in P \Rightarrow a \cdot b \in P$

Completeness Axiom: a least upper bound of a set A is a number x such that $x \geq y$ for all $y \in A$, and such that if z is also an upper bound for A, then necessarily $z \geq x$.

- **(P13)** (Existence of least upper bounds): Every nonempty set A of real numbers which is bounded above has a least upper bound.

We will call properties **(P1)**–**(P12)**, and anything that follows from them, *elementary arithmetic*. These properties imply, for example, that the real numbers contain the rational numbers as a subfield, and basic properties about the behavior of '>' and '<' under multiplication and addition.

Adding property **(P13)** uniquely determines the real numbers. The standard way of proving this is to identify each $x \in \mathbb{R}$ with the subset of rational numbers $y \in \mathbb{Q}$ such that $y \leq x$, referred to as a *Dedekind cut*. This procedure can also be used to construct the real numbers from the rationals.