Math 525, Winter 2008, Homework 2

The following homework is due Wednesday, January 30.

1. Show that the maximal function M is not bounded on $L^1(\mathbb{R}^n)$ by the following example. Let $f(x) = m(B(0,1))^{-1}\chi_{B(0,1)}$. Show that, for large $|x|$, then $Mf(x) \geq c|x|^{-n}$ some $c > 0$. Hence $\|Mf\|_{L^1} = \infty$.

Show that, even if you define $M'f = \sup_{0<r<1} A_r(|f|)$, then $M'f$ is not bounded on L^1, by considering $f(x) = m(B(0,\delta))^{-1}\chi_{B(0,\delta)}$. Show that $\|f_\delta\| = 1$, but $\|M'f_\delta\| \to \infty$ as $\delta \to 0$.

2. Suppose that g_δ is a family of positive, measurable functions on a measure space, which is decreasing in δ: for each point x, $g_\delta(x) \leq g_\eta(x)$ if $\delta \leq \eta$.

Suppose that, for all $c > 0$ and $\epsilon > 0$, there exists $\eta > 0$, such that

$$m(\{x : g_\delta(x) > c\}) < \epsilon \qquad \text{if} \qquad \delta < \eta$$

Show that $g_\delta(x) \to 0$ as $\delta \to 0$ for x a.e.

[Remark: this implies that if the weak-L^1 norm of g_δ goes to 0, with g_δ decreasing as above, then $g_\delta \to 0$ pointwise a.e.]

3. Problem 23, page 100 of Folland.

4. Problem 24, page 100 of Folland.

5. Problem 25, page 100 of Folland.