Math 525, Winter 2008, Homework 3

The **Midterm Exam** will be on Wednesday, February 13 or Friday, February 15.

The following homework is due Wednesday, February 6.

1. Let $\phi(x)$ be a bounded positive function on $\mathbb{R}^n$ with $\int \phi(x)\,dx = 1$. Set $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\varepsilon^{-1}x)$.

Suppose that there exist constants $c_j > 0$ with $\sum_{j=1}^\infty c_j = C < \infty$, and $r_j > 0$, such that

$$\phi(x) \leq \sum_{j=1}^{\infty} c_j \frac{1}{m(B(r_j,0))} \chi_{B(r_j,0)}\,.$$

Show that, if $f \in L^1(\mathbb{R}^n)$ and x is a Lebesgue point of f, then

$$\lim_{\varepsilon\to 0} \int \phi_\varepsilon(x-y)\, f(y)\, dy = f(x)\,.$$

[Hint: write

$$\left| f(x) - \int \phi_\varepsilon(x-y)\, f(y)\, dy \right| = \left| \int \phi_\varepsilon(x-y)\,[f(x)-f(y)]\, dy \right| \leq \int \phi_\varepsilon(x-y)\,|f(x)-f(y)|\, dy$$

then use the bound on ϕ, rescaled by ε and translated by y, to compare the last term to a sum of terms involving averages over balls centered at x. It helps to note that if $x \in L_f$ then $Mf(x) < \infty$.]

Remark: Any bounded, radial decreasing function ϕ with $\int \phi = 1$ can in fact be dominated in this way. The most important example is the Poisson kernel.

2. Problem 30, page 107 of Folland.

3. Problem 32, page 108 of Folland.

4. Problem 33, page 108 of Folland.

5. Problem 37, page 108 of Folland.