Sturm-Liouville Boundary Value Problems

Hart Smith

Department of Mathematics
University of Washington, Seattle

Math 526/556, Spring 2015
Sturm-Liouville operators

Second order differential operator on $[a, b]$:

$r \in C^1([a, b])$, $q \in C^0([a, b])$, both real-valued, and $r(x) > 0$.

$$Lu(x) = (r(x)u'(x))' + q(x)u(x)$$

$$= r(x)u''(x) + r'(x)u'(x) + q(x)u(x)$$

The operator $L$ is \textit{formally self-adjoint} in the following sense: if $u, v \in C^2([a, b])$, and both $= 0$ near the endpoints, then

$$\int_a^b \overline{u(x)} Lv(x) \, dx = \int_a^b \overline{u(x)} (r(x)v'(x))' + \overline{u(x)} q(x)v(x) \, dx$$

$$= \int_a^b -r(x) \overline{u'(x)} v'(x) + q(x) \overline{u(x)} v(x) \, dx$$

$$= \int_a^b \overline{Lu(x)} v(x) \, dx$$
Self-adjoint boundary conditions

For general $u, v \in C^2([a, b])$, maybe $\langle u, Lv \rangle \neq \langle Lu, v \rangle$:

$$\int_a^b \left( \overline{Lu} v - \overline{u} L v \right) \, dx = r(x) \left( \overline{u'}(x) v(x) - \overline{u(x)} v'(x) \right) \bigg|_{x=a}^{x=b}$$

Need conditions on $u, v$ at the boundary points $a$ and $b$ to ensure expression on right vanishes.

A boundary condition $Bu = 0$ is an equation, with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$,

$$Bu = \alpha u(a) + \beta u(b) + \gamma u'(a) + \delta u'(b) = 0$$

We will impose 2 linearly-independent boundary conditions

The pair of boundary conditions $B_1, B_2$ are self-adjoint for $L$ if

$$\int_a^b \overline{Lu(x)} v(x) \, dx = \int_a^b \overline{u(x)} L v(x) \, dx$$

for all $u, v$ that satisfy $B_1 u = B_1 v = 0$, $B_2 u = B_2 v = 0$. 
Self-adjoint boundary conditions

Examples:
- **Dirichlet**: $u(a) = 0$, $u(b) = 0$.

$u(a) = v(a) = 0$, $u(b) = v(b) = 0 \Rightarrow r(u'v - uv')|_{a}^{b} = 0$

- **Neumann**: $u'(a) = 0$, $u'(b) = 0$.

- **Robin**: $u'(a) = \alpha u(a)$, $u'(b) = -\beta u(b)$.

$u'(a)v(a) - u(a)v'(a) = \alpha u(a)v(a) - u(a)\alpha v(a) = 0$

- Any *separated* boundary conditions are self-adjoint:

$B_1 u = \alpha u(a) + \gamma u'(a) = 0$, $B_2 u = \beta u(b) + \delta u'(b) = 0$,\
A non-separated self-adjoint condition:

- **Periodic** (assume $r(a) = r(b)$): $u(a) = u(b), u'(a) = u'(b)$.

\[ r(a)(u'(a)v(a) - u(a)v'(a)) - r(b)(u'(b)v(b) - u(b)v'(b)) = 0 \]

**Notation**

Given $B_1, B_2$, let $C_B^2([a, b]) \subset C^2([a, b])$ denote the subspace of $u$ such that $B_1 u = B_2 u = 0$. So $B_1, B_2$ are self-adjoint for $L$ iff

\[ \langle u, Lv \rangle = \langle Lu, v \rangle \text{ for all } u, v \in C_B^2([a, b]). \]
Fix a strictly positive weight function \( \rho \in C^2([a, b]) \).

**Definition**

A Sturm-Liouville eigenvalue problem is the system

\[
Lu(x) = \lambda \rho(x) u(x) \quad \text{for} \quad x \in (a, b), \quad B_1 u = B_2 u = 0,
\]

where \( B_1, B_2 \) are self-adjoint for \( L \). The numbers \( \lambda \) for which there are non-zero solutions \( u \) are called eigenvalues, and the associated solutions \( u \) are eigenfunctions.

**Motivation**: Equation \( \rho(x) \partial_t^2 u(x, t) = Lu(x, t) \) has solutions

\[
w(x, t) = \sum_{\lambda} u_\lambda(x) \left( a_n \cos(\sqrt{-\lambda} t) + b_n \sin(\sqrt{-\lambda} t) \right)
\]

**Goal**: Show that the collection of eigenfunctions \( u_\lambda(x) \) form an orthonormal basis for \( L^2([a, b]) \).