VARIABLE COEFFICIENT NULL FORM ESTIMATES FOR $(1/2, 1/2)$ SYMBOLS

Hart F. Smith and Christopher D. Sogge

Abstract. We establish certain null-form estimates of Klainerman-Machedon for parametrices of variable coefficient wave equations for the convex obstacle problem, and for wave equations with metrics of bounded curvature.

1. Introduction

The purpose of this paper is to establish the following nullform estimate

$$\|Q(du,dv)\|_{H^1(\mathbb{R}^{1+3}_{t,x})} \le C\left(\|u_0\|_{H^2(\mathbb{R}^3)} + \|u_1\|_{H^1(\mathbb{R}^3)}\right)\left(\|v_0\|_{H^2(\mathbb{R}^3)} + \|v_1\|_{H^1(\mathbb{R}^3)}\right), \tag{1.1}$$

for solutions u and v to the Cauchy problem for the wave equation

$$\begin{aligned} \partial_t^2 u(t,x) &= \Delta_{\mathrm{g}} u(t,x)\,, \\ u(0,x) &= u_0(x)\,, \\ \partial_t u(0,x) &= u_1(x)\,. \end{aligned}$$

The null form Q may be any of of the following forms:

$$\begin{aligned} Q_0(du,dv) &= \partial_t u(t,x)\,\partial_t v(t,x) - \sum_{i,j=1}^{3} \mathrm{g}^{ij}(x)\partial_{x_i}u(t,x)\,\partial_{x_j}v(t,x)\,, \\ Q_{\alpha\beta}(du,dv) &= \partial_{x_\alpha}u(t,x)\,\partial_{x_\beta}v(t,x) - \partial_{x_\beta}u(t,x)\,\partial_{x_\alpha}v(t,x)\,, \end{aligned}$$

where x_α and x_β may represent t or any x_i .

For the Euclidean metric on $\mathbb{R}^3$, the estimate (1.1) was established globally by Klainerman and Machedon [KM]. For smooth, variable coefficient hyperbolic operators on an open ball, it was established by the second author in [So].

This paper is concerned with two new cases. The first is the case that the solutions u and v additionally satisfy Dirichlet conditions on a strictly geodesically concave hypersurface $\partial\Omega \subset \mathbb{R}^3$, that is,

$$u(t,x)\big|_{x\in\partial\Omega} = 0\,.$$

In this case we prove (1.1) for t in a small time interval and x in a small ball. In case Ω is the complement of a strictly convex obstacle in $\mathbb{R}^3$, with g the Euclidean metric, a partition of unity argument together with [KM] implies (1.1) globally in x, for t in any bounded interval.

Both authors are partially supported by the NSF.

The second is the case that $\mathbf{g}$ is a metric on a ball in $\mathbb{R}^3$, such that the components of the Riemann curvature tensor of $\mathbf{g}$ are bounded measurable functions. This is the notion of L^∞ pinched sectional curvature. For a precise statement of the conditions, we refer to [Sm2]. We then prove the estimates (1.1) for x and t in small sets.

In both cases, the problem is easily reduced to establishing the following estimate

$$\big\|Q(d\,Tf, d\,Tg)\big\|_{L^2(\mathbb{R}^{1+3}_{t,x})} \le C\,\|f\|_{H^1(\mathbb{R}^3)}\|g\|_{H^2(\mathbb{R}^3)}\,, \tag{1.2}$$

for an appropriate parametrix T of order 0. For the obstacle problem, the main part of the parametrix takes the form

$$Tf(t,x) = \sum_{\pm} \int e^{i\varphi^\pm(t,x,\xi)}\, a^\pm(t,x,\xi)\,\widehat{f}(\xi)\,d\xi\,,$$

where the phases $\varphi^\pm(t,x,\xi)$ satisfies the eikonal equation

$$\begin{aligned} |\partial_t\varphi^\pm(t,x,\xi)| \;&=\; \pm\|d_x\varphi(t,x,\xi)\|_{\mathbf{g}}\,, \\ \varphi^\pm(0,x,\xi) \;&=\; \langle x,\xi\rangle\,, \end{aligned} \tag{1.3}$$

and the symbols satisfy the following modified $S^0_{\frac23,\frac13}$ estimates

$$\big|\langle\xi,\partial_\xi\rangle^N\,\partial_x^\beta\partial_\xi^\alpha a^\pm(t,x,\xi)\big| \le C_{N,\alpha,\beta}\,\big(1+|\xi|\big)^{\frac{|\beta|}{3}-\frac{2|\alpha|}{3}}\,. \tag{1.3a}$$

There is also a "diffractive" term, whose estimation requires a simple modification of the argument for the main term, as will be shown in section 4.

In the case of the wave equation for metrics of bounded curvature tensor, the parametrix is more complicated. It takes the form

$$Tf(t,x) = \sum_{\pm}\sum_{k=0}^{\infty} \int e^{i\varphi_k^\pm(t,x,\xi)}\, a_k^\pm(t,x,\xi)\,\widehat{f}_k(\xi)\,d\xi\,,$$

where $\widehat{f}(\xi) = \sum_{k=0}^\infty \widehat{f}_k(\xi)$, and for $k\ge 1$ the support of $\widehat{f}_k(\xi)$ lies in $2^{k-1}\le|\xi|\le 2^{k+1}$.

The phases $\varphi_k^\pm$, which are homogeneous of degree 1 in ξ, satisfy the eikonal equation (1.3) for metrics $\mathbf{g}_k$, where $\mathbf{g}_k$ is a sequence of smooth metrics approximating the singular metric $\mathbf{g}$. This sequence of metrics satisfies the estimates

$$\big|\partial_x^\alpha \mathbf{g}_k(x)\big| \le \begin{cases} C\,, & |\alpha|\le 1\,,\\ C\,k\,, & |\alpha| = 2\,,\\ C_\alpha\,2^{k(|\alpha|-2)/2}\,, & |\alpha|\ge 3\,. \end{cases} \tag{1.4}$$

It also satisfies

$$\begin{aligned} \big|\,\mathbf{g}_k(x)-\mathbf{g}(x)\big| \;&\le\; C\,2^{-k}\,, \\ \big|\,\nabla_x\mathbf{g}_k(x)-\nabla_x\mathbf{g}(x)\big| \;&\le\; C\,2^{-\frac{k}{2}}\,. \end{aligned} \tag{1.5}$$

The sequence of phases satisfies corresponding estimates

$$\sup_{|\xi|=1}\big|\partial^\alpha_{t,x,\xi}\varphi_k(t,x,\xi)\big| \le \begin{cases} C\,, & |\alpha|\le 2\,,\\ C\,2^{k(|\alpha|-2)/2}\,, & |\alpha|\ge 2\,. \end{cases} \tag{1.6}$$

It also satisfies, for $m \ge k$,

$$\begin{aligned} \sup_{|\xi|=1} \bigl|\, \varphi_k^{\pm}(t,x,\xi) - \varphi_m^{\pm}(t,x,\xi) \bigr| \; &\le \; C\, 2^{-k}\,, \\ \sup_{|\xi|=1} \bigl|\, \nabla_{t,x,\xi}\varphi_k^{\pm}(t,x,\xi) - \nabla_{t,x,\xi}\varphi_m^{\pm}(t,x,\xi) \bigr| \; &\le \; C\, 2^{-\frac{k}{2}}\,. \end{aligned} \tag{1.7}$$

Finally, the symbols satisfy the following modified $S^0_{\frac12,\frac12}$ estimates

$$\bigl|\langle \xi, \partial_\xi\rangle^N \partial^\beta_{t,x} \partial^\alpha_\xi a_k^{\pm}(t,x,\xi)\bigr| \le C_{N,\alpha,\beta}\, 2^{k\left(\frac{|\beta|}{2}-\frac{|\alpha|}{2}\right)}\,. \tag{1.8}$$

The reduction of the estimate (1.1) to that of (1.2) is discussed for the obstacle problem in section 4 of this paper, and for the bounded curvature case in [Sm2]. The main work of this paper is to establish estimate (1.2) for parametrics of the above types. Since the main term of the parametrix associated to the convex obstacle problem is a particular case of that considered for bounded curvature metrics, we will consider the latter in the next two sections.

2. Further Reductions

We begin by reducing the proof of estimate (1.2) to consideration of the case that $\widehat{f}(\xi)$ is supported in a dyadic annulus at scale 2^k, and $\widehat{g}(\xi)$ is supported in a ball of radius $c\,2^k$, where one may choose c arbitrarily small. To do this, we write

$$Q(dTf, dTg) = \sum_k Q(dTf_k, dT\tilde g_k) + \sum_k Q(dT\tilde f_k, dTg_k)\,, \tag{2.1}$$

where

$$\begin{aligned} \operatorname{supp}\widehat{f_k}\,,\ \operatorname{supp}\widehat{g}_k \;&\subseteq\; \{\xi : 2^{k-1} \le |\xi| \le 2^{k+1}\}\,, \\ \operatorname{supp}\widehat{\tilde f_k} \;&\subseteq\; \{\xi : |\xi| \le 2^{k+N+1}\}\,, \\ \operatorname{supp}\widehat{\tilde g_k} \;&\subseteq\; \{\xi : |\xi| \le 2^{k-N}\}\,. \end{aligned}$$

Consider the second term of (2.1). By the Strichartz estimates, which hold for the parametrix T by [SmSo] and [Sm1, Sm2], we may bound

$$\begin{aligned} \sum_k \bigl\|Q(dT\tilde f_k, dTg_k)\bigr\|_{L^2(\mathbb{R}^{1+3}_{t,x})} \;&\le\; C \sum_k \|f_k\|_{H^{\frac32}(\mathbb{R}^3)}\, \|g_k\|_{H^{\frac32}(\mathbb{R}^3)} \\ &\le\; C \sum_k 2^{-\frac{k}{2}}\, \|f_k\|_{H^{\frac32}(\mathbb{R}^3)}\, \|g\|_{H^2(\mathbb{R}^3)} \\ &\le\; C\, \|f\|_{H^1(\mathbb{R}^3)}\, \|g\|_{H^2(\mathbb{R}^3)}\,. \end{aligned}$$

It thus remains to estimate the norm of the first term in (2.1). We first observe that, for N large enough, the terms are essentially mutually orthogonal over k. This follows by a simple integration by parts argument, which yields

$$\left|\int Q(dTf_k, dT\tilde g_k)\, \overline{Q(dTf_{k'}, dT\tilde g_{k'})}\, dt\, dx\right| \le C\, 2^{-|k-k'|}\, \|f_k\|\, \|g_k\|\, \|f_{k'}\|\, \|g_{k'}\|\,,$$

provided $|k-k'|\geq 3$, where $\|\cdot\|$ denotes the $L^2(\mathbb{R}^3)$ norm. Consequently,

$$\Big\|\sum_k Q(dTf_k, dT\tilde g_k)\Big\|^2_{L^2(\mathbb{R}^{1+3}_{t,x})} \leq C\sum_k \|\,Q(dTf_k, dT\tilde g_k)\|^2_{L^2(\mathbb{R}^{1+3}_{t,x})} + C\,\|f\|_{L^2(\mathbb{R}^3)}\,\|g\|_{L^2(\mathbb{R}^3)}\,.$$

Thus, to establish (1.2), it suffices to establish the following estimate, uniformly over k:

$$\|\,Q(dTf_k, dT\tilde g_k)\|_{L^2(\mathbb{R}^{1+3}_{t,x})} \leq C\,\|f_k\|_{H^1(\mathbb{R}^3)}\,\|\tilde g_k\|_{H^2(\mathbb{R}^3)}\,.$$

Finally, by the first estimate in (1.5), another application of the Strichartz estimates shows that we may replace the metric $\mathbf{g}$ in the form Q_0 by the metric $\mathbf{g}_k$.

By writing $T=T^++T^-$, there are essentially two terms to consider; $Q(dT^+f, dT^+g)$, and $Q(dT^+f, dT^-g)$. In what follows we consider the term $Q(dT^+f, dT^+g)$; the arguments hold with minor modification for the latter term. To simplify notation we use $\varphi_k(t,x,\xi)$ to denote $\varphi_k^+(t,x,\xi)$.

In the formula for the operator dT^+, the terms where d hits the symbol $a(t,x,\xi)$ are easily handled by the Strichartz and energy estimates; thus it suffices to restrict attention to the term where d falls on the phase. Let

$$q_{kj}(t,x,\xi,\eta) = Q\big(d\varphi_k(t,x,\xi/|\xi|), d\varphi_j(t,x,\eta/|\eta|)\big)\,.$$

The next reduction is to introduce polar coordinates for the η variable, $\eta=\rho\omega$, where $\rho\in\mathbb{R}^+$, and $\omega\in S^2$, the unit two-sphere. We introduce the operator

$$T^\omega(f,g) = \sum_{j\leq k-N}\int e^{i\varphi_k(t,x,\xi)+i\rho\varphi_j(t,x,\omega)} a_k(t,x,\xi)\, a_j(t,x,\rho\omega)\, q_{kj}(t,x,\xi,\omega)\, \widehat{f}_k(\xi)\, \widehat{g}_j(\rho)\, d\xi\, d\rho\,,$$

where $f\in L^2(\mathbb{R}^3)$ and $g\in L^2(\mathbb{R})$. A simple argument (see, e.g. [So]), now reduces the proof of (1.2) to showing that, for $g_\omega\in L^2(\mathbb{R}\times S^2)$, the following holds

$$\Big\|\int T^\omega(f,g_\omega)\,d\omega\Big\|_{L^2(dx\,dt)} \leq C\,\|f\|_{L^2(\mathbb{R}^3)}\,\|g_\omega\|_{L^2(\mathbb{R}\times S^2)}\,, \tag{2.2}$$

where the Fourier transforms of f and g_ω are restricted as above.

The next step, following Beals and Bezard [BB], is to reduce matters to the case that the null form takes values in a dyadic interval. To do this, let $Q(t,x,v,w)$ denote the quadractic form in $v=(v_t,v_x)$, $w=(w_t,w_x)$ associated to the null form. Let

$$\delta(l) = 2^{l-\frac{k}{4}}\,, \quad l\geq 0\,.$$

We write

$$Q(t,x,v,w) = \sum_{l=1}^{k} Q^l(t,x,v,w)\,,$$

where Q^l is the product of Q with a conic cutoff in v,w, which is supported in the region

$$\delta(l)\leq\left|\,\frac{v_x}{|v_x|}-\frac{w_x}{|w_x|}\,\right|\leq 4\,\delta(l)\,.$$

Let Q^0 be the remaining piece supported where the angle of v_x to w_x is bounded by $8\cdot 2^{-\frac{k}{4}}$. We correspondingly write

$$q_{kj}(t,x,\xi,\omega)=\sum_l q^l_{kj}(t,x,\xi,\omega)\,,$$

and $T^\omega=\sum_l T^{l,\omega}$, where

$$T^{l,\omega}(f,g)= \sum_{j\le k-N}\int e^{i\varphi_k(t,x,\xi)+i\rho\varphi_j(t,x,\omega)}a_k(t,x,\xi)\,a_j(t,x,\rho\omega)\,q^l_{kj}(t,x,\xi,\omega)\,\widehat{f}_k(\xi)\,\widehat{g}_j(\rho)\;d\xi\,d\rho\,. \tag{2.3}$$

By (1.6), and (1.4) in the case of the nullform Q_0 (recall that the metric $\mathbf{g}$ is replaced by $\mathbf{g}_k$), the following estimates are valid for $j\le k$:

$$\begin{aligned}\big|\langle\xi,\partial_\xi\rangle^N\partial^\beta_{t,x}\partial^\alpha_\xi q^l_{kj}(t,x,\xi,\omega)\big| &\le C_{N,\alpha,\beta}\,\delta(l)\,2^{\frac{k}{2}(|\beta|-|\alpha|)}\,,\\ \big|\langle\xi,\partial_\xi\rangle^N\partial^\beta_{t,x}\partial^\alpha_\xi\big(q^l_{kj}(t,x,\xi,\omega)-q^l_{kk}(t,x,\xi,\omega)\big)\big| &\le C_{N,\alpha,\beta}\,2^{-\frac{j}{2}}\,2^{\frac{k}{2}(|\beta|-|\alpha|)}\,.\end{aligned} \tag{2.4}$$

For fixed l, let

$$\widehat{f}(\xi)=\sum_\mu \widehat{f}_\mu(\xi)$$

be a decomposition where $\widehat{f}_\mu(\xi)$ is localised to a cone of angle $\delta(l)$.

Lemma 2.1. *For $l\ge 0$,*

$$\Big\|\sum_\mu\int T^{l,\omega}(f_\mu,g_\omega)\,d\omega\,\Big\|^2_{L^2(dx\,dt)}\le C\sum_\mu\Big\|\int T^{l,\omega}(f_\mu,g_\omega)\,d\omega\,\Big\|^2_{L^2(dx\,dt)}+C\,\|f\|_{L^2(\mathbb{R}^3)}\,\|g_\omega\|_{L^2(\mathbb{R}\times S^2)}\,.$$

Proof. We show that if $|\mu-\mu'|\ge C\,\delta(l)$, then

$$\int T^{l,\omega}(f_\mu,g)(t,x)\,\overline{T^{l,\omega}(f_{\mu'},g')(t,x)}\,dt\,dx\le C_N\,2^{-kN}\,\|f_\mu\|\,\|f_{\mu'}\|\,\|g\|\,\|g'\|\,, \tag{2.5}$$

where $\|\cdot\|$ denotes the $L^2(\mathbb{R}^3)$ norm. This follows by considering the operator $T^{l,\omega^*}T^{l,\omega'}$. On the region of integration it holds that the angle of $\nabla_x\varphi_k(t,x,\xi)+\rho\nabla_x\varphi_j(t,x,\omega)$ to $\nabla_x\varphi_k(t,x,\xi')+\rho'\nabla_x\varphi_j(t,x,\omega')$ is bounded below by $\delta(l)$, provided $|\mu-\mu'|\ge C\,\delta(l)$ for some large C. Thus

$$\big|\big(\nabla_x\varphi_k(t,x,\xi)+\rho\nabla_x\varphi_j(t,x,\omega)\big)-\big(\nabla_x\varphi_k(t,x,\xi')+\rho'\nabla_x\varphi_j(t,x,\omega')\big)\big|\ge c\,2^k\delta(l)\ge c\,2^{\frac{3k}{4}}\,.$$

An easy integration by parts in x yields (2.5). □

Lemma 2.2.

$$\Big\|\int T^{0,\omega}(f_\mu,g_\omega)\,d\omega\,\Big\|_{L^2(dx\,dt)}\le C\,\|f_\mu\|_{L^2(\mathbb{R}^3)}\,\|g_\omega\|_{L^2(\mathbb{R}\times S^2)}\,.$$

Proof. For fixed j and fixed (t,x), the function $T^{0,\omega}(f_\mu,g_{j\omega})(t,x)$ vanishes unless ω is in a set of volume $\delta(0)^2=2^{-\frac{k}{2}}$. Thus,

$$\Big\|\int T^{0,\omega}(f_\mu,g_\omega)\,d\omega\,\Big\|_{L^2(dx\,dt)}\le\sum_j 2^{-\frac{k}{4}}\;\big\|T^{0,\omega}(f_\mu,g_{j\omega})\big\|_{L^2(dx\,dt\,d\omega)}\,.$$

The operator

$$Af(x) = \int e^{i\varphi_k(t,x,\xi)} a_k(t,x,\xi)\, q^0_{kj}(t,x,\xi,\omega)\, \widehat{f}(\xi)\, d\xi \tag{2.6}$$

has L^2 norm, for each fixed t, less than $C\,\delta(0) = C\,2^{-\frac{k}{4}}$, with C independent of t (see e.g. [Sm2]). Thus

$$\| T^{0,\omega}(f_\mu, g_{j\omega})\|_{L^2(dx\,dt)} \le C\,2^{-\frac{k}{4}}\, \|f_\mu\|_{L^2(\mathbb{R}^3)}\, \|\widehat{g}_{j\omega}\|_{L^1(\mathbb{R})} \le C\,2^{-\frac{k}{4}}\, 2^{\frac{j}{2}}\, \|f_\mu\|_{L^2(\mathbb{R}^3)}\, \|g_{j\omega}\|_{L^2(\mathbb{R})}\,.$$

The result follows since

$$\sum_{j\le k} 2^{(j-k)/2} \|g_{j\omega}\|_{L^2(\mathbb{R}\times S^2)} \le \|g_\omega\|_{L^2(\mathbb{R}\times S^2)}\,. \quad \square$$

3. Nullform Estimates

In this section, we show that, for each fixed k, μ, and $l\ge 1$, the following holds:

$$\Big\| \int T^{l,\omega}(f_\mu, g_\omega)\, d\omega \Big\|_{L^2(dx\,dt)} \le C\,\delta(l)^{\frac14}\, |\log\delta(l)|^{\frac12}\, \|f_\mu\|_{L^2(\mathbb{R}^3)}\, \|g_\omega\|_{L^2(\mathbb{R}\times S^2)}\,, \tag{3.1}$$

with constant C independent of k, l, and μ. Together with Lemma 2.1 and Lemma 2.2, this implies estimate (2.2), which in turn implies the desired estimate (1.2).

We establish (3.1) by splitting the operator $T^{l,\omega}$ into two pieces. Let

$$\begin{aligned} T_0^{l,\omega}(f,g) &= \sum_j T^{l,\omega}(f, g_j) \ :\ 2^j \le 2^{\frac{k}{2}}\,\delta(l)^{-1}\,, \\ T_1^{l,\omega}(f,g) &= \sum_j T^{l,\omega}(f, g_j) \ :\ 2^j > 2^{\frac{k}{2}}\,\delta(l)^{-1}\,. \end{aligned} \tag{3.2}$$

For the operator $T_1^{l,\omega}$, it holds that $2^{-\frac{j}{2}} \le \delta(l)$, where we use the fact that $\delta(l)\ge 2^{-\frac{k}{4}}$. Hence, the integrand vanishes unless

$$\text{angle}\,\big(\, d_x\varphi_k(t,x,\xi/|\xi|)\,,\, d_x\varphi_k(t,x,\omega)\,\big) \le C\,\delta(l)\,.$$

Since the map $\xi \to d_x\varphi_k(t,x,\xi)/|d_x\varphi_k(t,x,\xi)|$ is a C^1 diffeomorphism of the unit sphere, with uniform bounds over t, x, and k, for t small, it follows that the integrand vanishes unless $\big|\mu-\omega\big| \le C\,\delta(l)$. Consequently,

$$\Big\| \int T_1^{l,\omega}(f_\mu, g_\omega)\, d\omega \,\Big\|_{L^2(dx\,dt)} \le \delta(l)\, \big\| T_1^{l,\omega}(f_\mu, g_\omega)\,\big\|_{L^2(dx\,dt\,d\omega)}\,.$$

For the piece $T_1^{l,\omega}$, the estimate (3.1) is thus implied by the following

Theorem 3.1. *The following holds*

$$\big\| T_1^{l,\omega}(f,g)\big\|_{L^2(dx\,dt)} \le C\,\delta(l)^{-\frac34}\, |\log\delta(l)|^{\frac12}\, \|f\|_{L^2(\mathbb{R}^3)}\, \|g\|_{L^2(\mathbb{R})}\,.$$

We postpone the proof of Theorem 3.1, and first establish the following.

Theorem 3.2. *The following holds*

$$\Big\| \int T_0^{l,\omega}(f_\mu, g_\omega)\, d\omega \Big\|_{L^2(dx\,dt)} \le C\,\delta(l)\,|\log\delta(l)|\,\|f_\mu\|_{L^2(\mathbb{R}^3)}\,\|g_\omega\|_{L^2(\mathbb{R}\times S^2)}\,.$$

Proof. We split the sum over j in (3.2) into three disjoint cases.

Case 1: $2^j \le \delta(l)^{-2}$. In this case the index j runs over $|\log\delta(l)|$ values. Also, for fixed j and fixed (t,x), the integrand vanishes unless ω lies in a set in S^2 of area $\delta(l)^2$. Hence it suffices to establish the following estimate, uniformly in j and ω:

$$\big\|T^{l,\omega}(f, g_j)\big\|_{L^2(dx\,dt)} \le C\,\|f\|_{L^2(\mathbb{R}^3)}\,\|g_j\|_{L^2(\mathbb{R})}\,. \tag{3.3}$$

We write $\widehat{f}(\xi) = \sum_\nu \widehat{f}_\nu(\xi)$ where $\widehat{f}_\nu$ is supported in a cone of angle $2^{-\frac{k}{2}}$ about the unit vector ν. The estimate (3.3) is a result of the following.

$$\Big| \int T^{l,\omega}(f_\nu, g_j)\,\overline{T^{l,\omega}(f_{\nu'}, g_j)}\,dt\,dx \Big| \le C\,\big(1 + 2^{\frac{k}{2}}|\,\nu-\nu'|\big)^{-N}\,\|f_\nu\|_{L^2(\mathbb{R}^3)}\,\|f_{\nu'}\|_{L^2(\mathbb{R}^3)}\,\|g_j\|^2_{L^2(\mathbb{R})}\,. \tag{3.4}$$

The (t,x) integrand in (3.4) takes the form

$$\delta(l)^2 \int e^{i\varphi_k(t,x,\xi) - i\varphi_k(t,x,\xi')}\,a_{\nu,\nu'}(t,x,\xi,\xi')\,\widehat{f}_\nu(\xi)\,\widehat{f}_{\nu'}(\xi')\,d\xi\,d\xi' \times \Big| \int e^{i\rho\varphi_j(t,x,\omega)}\,a_j(t,x,\rho\omega)\,\widehat{g}_j(\rho)\,d\rho \Big|^2, \tag{3.5}$$

where, by (1.8) and (2.4),

$$\Big| \partial^\beta_{t,x}\partial^\alpha_{\xi,\xi'}\langle \nu, \partial_\xi\rangle^m \langle \nu', \partial_{\xi'}\rangle^{m'} a_{\nu,\nu'}(t,x,\xi,\xi') \Big| \le C\,2^{\frac{k}{2}(|\beta|-|\alpha|) - k(m+m')}\,. \tag{3.6}$$

Since $\rho \le \delta(l)^{-2} \le 2^{-\frac{k}{2}}$, the operator $2^{-\frac{k}{2}}\partial_x$ applied to the expression inside the absolute value sign in (3.5) leads to an expression of the same form. Furthermore, on the (ξ,ξ') support of the symbol in (3.5), the following holds,

$$2^{-\frac{k}{2}}\big|\partial_x\varphi_k(t,x,\xi) - \partial_x\varphi_k(t,x,\xi')\big| \ge 2^{\frac{k}{2}}\,|\,\nu - \nu'|\,.$$

Integration by parts in x now bounds the left hand side of (3.4) by

$$\frac{\delta(l)^2}{\big(1 + 2^{\frac{k}{2}}|\,\nu-\nu'|\big)^N} \int \Big| \int e^{i\varphi_k(t,x,\xi) - i\varphi_k(t,x,\xi')}\,a_{\nu,\nu'}(t,x,\xi,\xi')\,\widehat{f}_\nu(\xi)\,\widehat{f}_{\nu'}(\xi')\,d\xi\,d\xi' \Big| \times \Big| \int e^{i\rho\varphi_j(t,x,\omega)}\,a_j(t,x,\rho\omega)\,\widehat{g}_j(\rho)\,d\rho \Big|^2 dt\,dx\,,$$

where $a_{\nu,\nu'}(t,x,\xi,\xi')$ is a symbol satisfying the same estimates (3.6). Next, following [SSS], we replace the phase $\varphi_k(t,x,\xi)$ by $\langle \nabla_x\varphi_k(t,x,\nu), \xi\rangle$, modulo an error that is absorbed into the symbol, and similarly for $\varphi_k(t,x,\xi')$. The left hand side of (3.4) is thus bounded by

$$\frac{\delta(l)^2}{\big(1 + 2^{\frac{k}{2}}|\,\nu-\nu'|\big)^N} \int f^*_\nu\big(\nabla_x\varphi_k(t,x,\nu)\big)\,f^*_{\nu'}\big(\nabla_x\varphi_k(t,x,\nu')\big)\,g^*_j\big(\varphi_j(t,x,\omega)\big)^2\,dt\,dx\,, \tag{3.7}$$

where

$$f_\nu^*(y) = 2^{2k} \int \bigl(1 + 2^{\frac{k}{2}}|y-z| + 2^k|\langle \nu, y-z\rangle|\bigr)^{-4} |f_\nu(z)|\, dz\,,$$

and

$$g_j^*(s) = \int \bigl(1 + 2^j|s-r|\bigr)^{-2} |g_j(r)|\, ds\,,$$

hence

$$\|f_\nu^*\|_{L^2(dy)} \le \|f_\nu\|_{L^2(\mathbb{R}^3)}\,, \quad \|g_j^*\|_{L^2(ds)} \le \|g_j\|_{L^2(\mathbb{R})}\,.$$

The change of variables $(t,x) \to \bigl(\varphi_j(t,x,\omega), \nabla_x\varphi_k(t,x,\nu)\bigr)$ has Jacobian comparable to $\delta(l)$. An application of the Schwartz inequality to (3.7) yields (3.4).

Case 2: $\delta(l)^{-2} \le 2^j \le 2^{\frac{k}{2}}$. Consider the operator $\tilde T^{l,\omega}$ obtained by replacing $q^l_{jk}(t,x,\xi,\omega)$ in equation (2.3) by $q^l_{jk}(t,x,\xi,\omega) - q^l_{kk}(t,x,\xi,\omega)$. The proof of the previous case, together with the second set of estimates in (2.4), shows that for $2^j \le 2^{\frac{k}{2}}$ the following holds,

$$\Bigl\| \int \tilde T^{l,\omega}(f_\mu, g_{j\omega})\, d\omega \Bigr\|_{L^2(dx\,dt)} \le C\, 2^{-\frac{j}{2}}\, \|f_\mu\|_{L^2(\mathbb{R}^3)}\, \|g_{j\omega}\|_{L^2(\mathbb{R}\times S^2)}\,.$$

Summing over j such that $2^j \ge \delta(l)^{-2}$ yields the estimate of Theorem 3.2 for this case if $T^{l,\omega}$ is replaced by $\tilde T^{l,\omega}$. It thus remains to establish the same estimate for the term

$$S^{l,\omega}(f_\mu, g_\omega) = \Bigl(\int e^{i\varphi_k(t,x,\xi)}\, a_k(t,x,\xi)\, q^l_{kk}(t,x,\xi,\omega)\, \widehat f_\mu(\xi)\, d\xi\Bigr) \sum_j e^{i\rho\varphi_j(t,x,\omega)}\, a_j(t,x,\rho\omega)\, \widehat g_j(\rho)\, d\rho\,.$$

The ξ integrand vanishes unless $|\omega - \mu| \le \delta(l)$, hence

$$\Bigl\| \int S^{l,\omega}(f_\mu, g_\omega)\, d\omega \Bigr\|_{L^2(dx\,dt)} \le \delta(l)\, \bigl\| S^{l,\omega}(f_\mu, g_\omega) \bigr\|_{L^2(dx\,dt\,d\omega)}\,.$$

The proof of estimate (3.4) establishes the following bound,

$$\Bigl| \int S^{l,\omega}(f_\nu, g)\, \overline{S^{l,\omega}(f_{\nu'}, g)}\, dt\, dx \Bigr|$$
$$\le C\bigl(1 + 2^{\frac{k}{2}}|\nu - \nu'|\bigr)^{-N} \|f_\nu\|_{L^2(\mathbb{R}^3)}\, \|f_{\nu'}\|_{L^2(\mathbb{R}^3)} \Bigl(\sup_y \|Pg\|_{L^2(ds)}\Bigr) \Bigl(\sup_{y'} \|P'g\|_{L^2(ds')}\Bigr)\,,$$

where Pg is an operator of the form

$$Pg = \sum_j e^{i\rho\varphi_j(t,x,\omega)}\, a_j(t,x,\rho\omega)\, \widehat g_j(\rho)\, d\rho\,,$$

written in the new coordinates

$$(s,y) = \bigl(\varphi_k(t,x,\omega), \nabla_x\varphi_k(t,x,\nu)\bigr)\,,$$

and $P'g$ is the same form with ν replaced by ν'. Using (1.7) we may write Pg in the form

$$Pg(s,y) = \sum_j \int e^{is\rho}\, a_j(s,y,\rho)\, \widehat g_j(\rho)\, d\rho\,,$$

where the symbol satisfies

$$\left|\partial_s^m \partial_\rho^n a_j(s,y,\rho)\right| \le C\left(2^{\frac{j}{2}}\delta(l)^{-2}\right)^m 2^{-jn}\,, \quad m \le 1\,.$$

A simple integration by parts establishes the following bound,

$$\left|\int e^{is(\rho-\rho')}\, a_j(s,y,\rho)\,\widehat{g}_j(\rho)\,\overline{a_{j'}(s,y,\rho')\,\widehat{g}_{j'}(\rho')}\,d\rho\,d\rho'\,ds\right| \tag{3.8}$$
$$\le C\left(1+\delta(l)^2\,2^{\max(j,j')/2}\right)^{-1} \|g_j\|_{L^2(\mathbb{R})}\|g_{j'}\|_{L^2(\mathbb{R})}\,.$$

Summing over j yields the following,

$$\sup_y \|Pg\|_{L^2(ds)} \le C\,|\log\delta(l)|\,\|g\|_{L^2(\mathbb{R})}\,,$$

which completes the proof for the second case.

Case 3: $2^{\frac{k}{2}} \le 2^j \le 2^{\frac{k}{2}}\delta(l)^{-1}$. There are $|\log\delta(l)|$ terms j, so as in the first case it suffices to establish the estimate (3.4) uniformly over j. Let

$$v(t,x) = \frac{1}{|\nu-\nu'|}\times\left(\text{projection of } \nabla_x\varphi_k(t,x,\nu) - \nabla_x\varphi_k(t,x,\nu') \text{ onto } \nabla_x\varphi_k(t,x,\nu)^\perp\right)\,.$$

Since $|\nu-\nu'| \ge 2^{-\frac{k}{2}}$, it follows from (1.6) that

$$\left|\partial_{t,x}^\alpha v(t,x)\right| \le C_\alpha\, 2^{\frac{k}{2}|\alpha|}\,.$$

Next note that, since $\left|\nu - \xi/|\xi|\right| \le 2^{-k/2} \le \delta(l)$, then on the support of $q_{kj}^l(t,x,\xi,\omega)$ the following holds,

$$\left|\partial_{t,x}^\alpha\langle v(t,x),\nabla_x\varphi_j(t,x,\omega)\rangle\right| = \left|\partial_{t,x}^\alpha\langle v(t,x),\nabla_x\varphi_j(t,x,\omega)-\nabla_x\varphi_k(t,x,\nu)\rangle\right| \le C_\alpha\,\delta(l)\,2^{\frac{k}{2}|\alpha|}\,.$$

Since $\rho\,\delta(l) \le 2^{\frac{k}{2}}$, it follows that one may write

$$\left(2^{-\frac{k}{2}}\langle v(t,x),\nabla_x\rangle\right)^N\left(\int e^{i\rho\varphi_j(t,x,\omega)}\,a_j(t,x,\rho\omega)\,\widehat{g}_j(\rho)\,d\rho\right)$$

as an expression of the same form, but with a new symbol which satisfies the estimates

$$\left|\partial_{t,x}^\alpha\partial_\rho^m\tilde{a}_j(t,x,\rho)\right| \le C_{\alpha,m}\,2^{\frac{k}{2}|\alpha|-mj}\,.$$

With such a symbol one still has the estimates

$$\left|\int e^{i\rho\varphi_j(t,x,\omega)}\,\tilde{a}_j(t,x,\rho\omega)\,\widehat{g}_j(\rho)\,d\rho\right| \le g_j^*(\varphi_j(t,x,\omega))\,.$$

We next note that

$$\langle v(t,x),\nabla_x\varphi_k(t,x,\xi)-\nabla_x\varphi_k(t,x,\xi')\rangle \approx 2^k\,|\nu-\nu'|\,.$$

The proof in the first case of estimate (3.4) now carries over to the third case, where in establishing estimate (3.7), one integrates by parts with respect to

$$\frac{1 - i\,2^{-\frac{k}{2}}\langle v(t,x),\nabla_x\rangle}{1+2^{-\frac{k}{2}}\langle v(t,x),\nabla_x\varphi_k(t,x,\xi)-\nabla_x\varphi_k(t,x,\xi')\rangle}\,. \quad \square$$

The proof of Theorem 3.1 rests on the following two lemmas estimating the gradients of the phase function.

Lemma 3.3. *Let $\Phi(t,x,t',x',\xi)=\varphi(t,x,\xi)-\varphi(t',x',\xi)$, where $\varphi=\varphi_k$ for some k. Suppose that ω is a unit vectors, and let $\delta=\big|\omega-\xi/|\xi|\big|$. If $|\xi|=2^k$, then for some $c>0$,*

$$2^{\frac{k}{2}}|\nabla_\xi\Phi(t,x,t',x',\xi)|+|\Phi(t,x,t',x',\xi)|+2^j|\Phi(t,x,t',x',\omega)| \geq c\Big(2^{\frac{k}{2}}\,|\nabla_\xi\Phi(t,x,t',x',\omega)|+2^{\frac{k}{2}}\,\delta\,|t-t'|\Big)\,, \tag{3.9}$$

for all j, k and δ such that $j\leq k$, $2^j\,\delta\geq 2^{\frac{k}{2}}$, and $\delta\geq 2^{-\frac{k}{2}}$.

Proof. We have $2^k\geq 2^j\geq 2^{\frac{k}{2}}\delta^{-1}$, so introducing the new coordinates

$$y_j(t,x)=\partial_{\xi_j}\varphi(t,x,\xi)\,,$$

the left hand side of (3.9) is larger than

$$2^{\frac{k}{2}}|y-y'|+2^{\frac{k}{2}}\delta^{-1}\Big(|\langle\mu,y-y'\rangle|+|F(t,y)-F(t',y')|\Big)\,, \tag{3.10}$$

where $F(t,y)$ is $\varphi(t,x,\omega)$ written in the coordinates (t,y), and $\mu=\xi/|\xi|$. We begin by showing that the quantity (3.10) is larger than

$$2^{\frac{k}{2}}\big(\,|y-y'|+\delta\,|t-t'|\,\big)\,.$$

To see this, note that the C^1 distance of $F(t,y)$ to $\langle\mu,y\rangle$ is of size δ, so that

$$F(t',y)-F(t',y')=\langle\mu,y-y'\rangle+O\big(\delta|y-y'|\big)\,.$$

Thus (3.10) dominates

$$2^{\frac{k}{2}}\big(\,|y-y'|+\delta^{-1}|F(t,y)-F(t',y)|\,\big)\,.$$

We will be done by showing that

$$\partial_t F(t,y)=\|d_x\varphi(t,x,\omega)\|_{\mathbf{g}}-\mathbf{g}\big(d_x\varphi(t,x,\omega),d_x\varphi(t,x,\mu)\big)/\|d_x\varphi(t,x,\mu)\|_{\mathbf{g}}\approx\delta^2\,,$$

where $\mathbf{g}=\mathbf{g}_k$. To see this, let $x(t,y)$ denote x in the (t,y) coordinates. Then since $x(t,y)$ is the backwards geodesic through (y,μ), the eikonal equations give

$$\partial_t x_i=-\mathbf{g}^{ki}(t,x)\partial_{x_k}\varphi(t,x,\mu)/\|d_x\varphi(t,x,\mu)\|_{\mathbf{g}}\,.$$

Thus,

$$\begin{aligned}\partial_t F(t,y)&=\partial_t\varphi(t,x(t,y),\omega)\\&=\partial_t\varphi(t,x,\omega)+\partial_{x_i}\varphi(t,x,\omega)\partial_t x_i\\&=\|d_x\varphi(t,x,\omega)\|_{\mathbf{g}}-\mathbf{g}\big(d_x\varphi(t,x,\omega),d_x\varphi(t,x,\mu)\big)/\|d_x\varphi(t,x,\mu)\|_{\mathbf{g}}\,.\end{aligned}$$

To finish the proof of the lemma, let $f_j(t,y)$ denote $\partial_{\xi_j}\varphi(t,x,\omega)$ in the (t,y) coordinates. Then the C^1 distance of f_j to y_j is comparable to δ, so that $|\partial_t f(t,y)|\leq\delta$. Consequently,

$$\big|\nabla_\xi\Phi(t,x,t',x',\omega)\big|=|f_j(t,y)-f_j(t',y')|\leq C\big(\,|y-y'|+\delta\,|t-t'|\,\big)\,.\quad\square$$

Lemma 3.4. *Let $\Phi(t,x,\xi,\xi') = \varphi(t,x,\xi) - \varphi(t,x,\xi')$, where $\varphi = \varphi_k$ for some k. Suppose that*

$$C^{-1}\delta \quad \leq \quad \left| \omega - \frac{\xi}{|\xi|} \right| \quad , \quad \left| \omega - \frac{\xi'}{|\xi'|} \right| \quad \leq \quad C\delta\,,$$

that $|\xi|\,,\,|\xi'| \in \left[2^{k-1}, 2^{k+1}\right]$, and that $|\rho|\,,\,|\rho'| \in [0, 2^{k-N}]$. Then for some $c > 0$, independent of k,δ,

$$\left| \nabla_{t,x}\Phi(t,x,\xi,\xi') + \nabla_{t,x}\Phi(t,x,\rho\omega,\rho'\omega) \right| \geq c\left(2^k\,\delta\,|\,\nu - \nu'\,| + \delta^2\,|\rho - \rho'\,| \right). \tag{3.11}$$

Proof. Let $w = \nabla_x\varphi(t,x,\xi)$, $w' = \nabla_x\varphi(t,x,\xi')$, and $\mu = \nabla_x\varphi(t,x,\omega)$. Also let $\alpha = \rho - \rho'$. The conditions of the statement imply that the angle of w or w' to μ is comparable to δ.

By the eikonal equations, the left hand side of (3.11) dominates

$$\left|w - w' + \alpha\mu\right| + \left|\,\|w\| - \|w'\| + \alpha\|\mu\|\,\right| \tag{3.12}$$

where $\|\cdot\|$ denotes the norm in the metric $\mathbf{g}_k$. We consider the case $\alpha \geq 0$; the case $\alpha \leq 0$ follows by symmetry after exchanging w and w'. Also, by scaling α, we may assume that $\|\mu\| = 1$, and that after scaling $\alpha \leq \frac{1}{2}\|w\|$ (for N sufficiently large.) The quantity (3.12) dominates

$$\begin{aligned}
\|w\| + \alpha - \|w + \alpha\mu\| &\geq c\,2^{-k}\left(\left(\|w\| + \alpha\right)^2 - \left(\|w + \alpha\mu\|\,\right)^2 \right) \\
&\geq c\,2^{-k}\,\|w\|\,\alpha\,\left\| \frac{w}{\|w\|} - \mu \right\|^2 \\
&\geq c\,\delta^2\,\alpha\,.
\end{aligned}$$

We next observe that (3.12) dominates the following quantity

$$\left\| \frac{w}{\|w\|} - \mu - r\left(\frac{w'}{\|w'\|} - \mu \right) \right\|, \tag{3.13}$$

where $r = \|w'\|/\|w\| \in [c, c^{-1}]$. By making a linear transformation, we may replace the $\mathbf{g}$ norm $\|\cdot\|$ by the Euclidean norm $|\cdot|$. We set

$$\mu = (1,0,0)\,, \qquad \frac{w}{|w|} = (\cos\theta, \sin\theta, 0)\,, \qquad \frac{w'}{|w'|} = (\sqrt{1-z^2}\cos\gamma, \sqrt{1-z^2}\sin\gamma, z)\,,$$

where θ,γ,z are small. The quantity (3.13) is then comparable to

$$2^k\left(\left|\,1 - \cos\theta - r\,(1 - \sqrt{1-z^2}\cos\gamma)\,\right| + \left|\,\sin\theta - r\sqrt{1-z^2}\sin\gamma\,\right| + |z| \right),$$

which in turn is comparable to

$$2^k\left(\left|\,1 - \cos\theta - r\,(1 - \cos\gamma)\,\right| + \left|\,\sin\theta - r\sin\gamma\,\right| + |z| \right).$$

By the half angle formula, this equals

$$2^k\left(\left|\,\sin\theta\,\tan\frac{\theta}{2} - 2r\sin^2\frac{\gamma}{2}\,\right| + \left|\,\sin\theta - 2r\sin\frac{\gamma}{2}\,\cos\frac{\gamma}{2}\,\right| + |z| \right).$$

Since $\cos(\gamma/2) \approx 1$, this in turn dominates

$$2^k \Big(|\sin\theta| \, \big|\tan\frac{\theta}{2} - \tan\frac{\gamma}{2}\big| + |z| \Big) \ge c\, 2^k \, \big(\delta\,|\theta-\gamma| + |z|\big) \ge c\, 2^k \, \delta\,|\nu - \nu'|\,. \quad \square$$

Proof of Theorem 3.1. To complete the proof, we make a further decomposition

$$T^{l,\omega} = \sum_{\nu,j,s,z} T^{l,\omega}_{\nu,j,s,z}\,,$$

where

$$T^{l,\omega}_{\nu,j,s,z} F(t,x) = \int e^{i\varphi_k(t,x,\xi) + i\rho\varphi_k(t,x,\omega)} a^{l,\omega}_{\nu,j,s,z}(t,x,\xi,\rho)\,\widehat{F}(\xi,\rho)\,d\xi\,d\rho\,. \tag{3.14}$$

The index ν varies over a set of unit vectors evenly spaced by $c\,2^{-\frac{k}{2}}\,\delta(l)^{-1}$, for some small c to be determined independent of k and l. The index j runs over the integers such that

$$2^k \;\ge\; 2^j \;\ge\; 2^{\frac{k}{2}}\,\delta(l)^{-1}\,.$$

The indices z and s run over lattices such that

$$2^{\frac{k}{2}} z \in \mathbb{Z}^3\,, \qquad 2^{\frac{k}{2}}\,\delta(l)\, s \in \mathbb{Z}\,.$$

The symbol $a^{l,\omega}_{\nu,j,s,z}(t,x,\xi,\rho)$ is supported in the set where $|\xi| \in [2^{k-1}, 2^{k+1}]$, $\quad \rho \in [2^{j-1}, 2^{j+1}]$, and where

$$\Big|\,\frac{\xi}{|\xi|} - \nu\,\Big| \le c\,2^{-\frac{k}{2}}\,\delta(l)^{-1}\,, \qquad \big|\,\nabla_\xi\varphi_k(t,x,\omega) - z\,\big| \le 2\cdot 2^{-\frac{k}{2}}\,, \qquad |\,t-s\,| \le 2\cdot 2^{-\frac{k}{2}}\,\delta(l)^{-1}\,,$$

The symbol furthermore satisfies the estimates

$$\big|\partial_t^p \partial_x^\beta \partial_\xi^\alpha \partial_\rho^m \langle \xi, \partial_\xi\rangle^i a^{l,\omega}_{\nu,j,s,z}(t,x,\xi,\rho)\big| \le C\,\delta(l)\,2^{-jm - \frac{k}{2}(p+|\beta|-|\alpha|)}\,. \tag{3.15}$$

A few remarks are in order here. First, the function

$$e^{i\rho\varphi_j(t,x,\omega) - i\rho\varphi_k(t,x,\omega)}$$

satisfies the symbol estimates (3.15), which allowed us to replace the phase $\varphi_j(t,x,\omega)$ by $\varphi_k(t,x,\omega)$ in formula (3.14). Next, since

$$2^{-\frac{j}{2}} \le 2^{-\frac{k}{8}}\,\delta(l) \ll \delta(l)\,,$$

it follows from (1.7) and the definition of q^l_{kj} that the angle of $\nabla_x\varphi_k(t,x,\omega)$ to $\nabla_x\varphi_k(t,x,\xi)$ is comparable to $\delta(l)$, hence that the angle of ω to ξ is comparable to $\delta(l)$. By making the number c above small, and recalling that $\delta(l) \ge 2^{-\frac{k}{4}}$, it follows that the angle of ω to ν is comparable to $\delta(l)$.

We begin by showing that

$$\sup_{s',z'} \sum_{s,z} \big\|\, T^{l,\omega}_{\nu,j,s,z}\big(T^{l,\omega}_{\nu,j,s',z'}\big)^* \,\big\|^{\frac{1}{2}} \le C\,\delta(l)^{-1}\,. \tag{3.16}$$

This will follow from showing that

$$\| T^{l,\omega}_{\nu,j,s,z}\big(T^{l,\omega}_{\nu,j,s',z'}\big)^* \| \;\le\; \frac{C_N\,\delta(l)^{-2}}{\big(1+2^{\frac k2}|\,z-z'|+2^{\frac k2}\,\delta(l)\,|\,s-s'|\big)^N}\,. \tag{3.17}$$

To establish (3.17), we express $T^{l,\omega}_{\nu,j,s,z}\big(T^{l,\omega}_{\nu,j,s',z'}\big)^*$ as an integral kernel of the form

$$K(t,x;t',x')=\int e^{i\varphi_k(t,x,\xi)-i\varphi_k(t',x',\xi)+i\rho(\varphi_k(t,x,\omega)-\varphi_k(t',x',\omega))}$$
$$a^{l,\omega}_{\nu,j,s,z}(t,x,\xi,\rho)\,\overline{a}^{l,\omega}_{\nu,j,s',z'}(t',x',\xi,\rho)\,d\xi\,d\rho\,.$$

Integration by parts in ξ and ρ, together with the estimates (3.15) and the support conditions, shows that the kernel $|K(t,x;t',x')|$ is bounded by

$$\int_{R_{\nu,j}}\frac{C_N\,\delta(l)^2}{\big(1+2^{\frac k2}\big|\,\nabla_\xi\Phi(t,x,t',x',\xi)\,\big|+\big|\,\Phi(t,x,t',x',\xi)\,\big|+2^j\big|\,\Phi(t,x,t',x',\omega)\,\big|\big)^N}\,d\xi\,d\rho\,,$$

where Φ is as in Lemma 3.3. For each ξ in the domain of integration, the change of variables $(t,x)\to\big(\nabla_\xi\varphi_k(t,x,\xi),\varphi_k(t,x,\omega)\big)$ has Jacobian comparable to $\delta(l)^2$; consequently, by Schur's Lemma and Lemma 3.3, for each fixed ξ and ρ the integrand is an operator on $L^2(dt\,dx)$ with norm bounded by

$$\frac{C_N\,2^{-2k-j}}{\big(1+2^{\frac k2}|\,z-z'|+2^{\frac k2}\,\delta(l)\,|\,s-s'|\big)^N}\,.$$

The volume of $R_{\nu,j}$ is comparable to $2^{2k+j}\,\delta(l)^{-2}$, and the estimate (3.17) follows.

We next establish the following estimate

$$\sup_{j',\nu'}\sum_{j,\nu}\|\,\big(T^{l,\omega}_{\nu,j,s,z}\big)^*\,T^{l,\omega}_{\nu',j',s,z}\,\|^{\frac12}\;\le\;C\,\delta(l)^{-\frac12}\,|\log\delta(l)|\,. \tag{3.18}$$

This will follow from showing that, for $|j-j'|\ge 2$,

$$\|\,\big(T^{l,\omega}_{\nu,j,s,z}\big)^*\,T^{l,\omega}_{\nu',j',s,z}\,\|\;\le\;\frac{C_N\,\delta(l)^{-1}}{\big(1+2^{-\frac k2}\,\delta(l)^2\,\big|2^j-2^{j'}\big|+2^{\frac k2}\,\delta(l)\,|\,\nu-\nu'|\big)^N}\,. \tag{3.19}$$

That (3.18) is a result of (3.19) follows from that fact that

$$\sum_j\frac{1}{1+2^{-\frac k2}\,\delta(l)^2\,|\,2^j-2^{j'}|}\le C\,|\log\delta(l)|\,,$$

where the sum is over j such that $2^j\ge 2^{\frac k2}\,\delta(l)^{-1}$.

To establish (3.19), we note that $\big(T^{l,\omega}_{\nu,j,s,z}\big)^*T^{l,\omega}_{\nu',j',s,z}$ has an integral kernel of the form

$$K(\xi,\rho;\xi',\rho')=\int e^{i\varphi_k(t,x,\xi)-i\varphi_k(t,x,\xi')+i\varphi_k(t,x,\rho\omega)-i\varphi_k(t,x,\rho'\omega)}$$
$$\overline{a}^{l,\omega}_{\nu,j,s,z}(t,x,\xi,\rho)\,a^{l,\omega}_{\nu',j',s,z}(t,x,\xi',\rho')\,dt\,dx\,.$$

Integration by parts in (t, x) yields the following bound,

$$|K(\xi,\rho;\xi',\rho')| \le \int_{R_{s,z}} \frac{C_N\,\delta(l)^2}{\Big(1+2^{-\frac{k}{2}}\Big|\,\nabla_{t,x}\Phi(t,x,\xi,\xi')+\nabla_{t,x}\Phi(t,x,\rho\omega,\rho'\omega)\,\Big|\Big)^N}\,dt\,dx\,,$$

where Φ is as in Lemma 3.4, and $R_{s,z}$ is a set of volume $2^{-2k}\delta(l)^{-1}$. The change of variables $(\xi,\rho) \to \nabla_{t,x}\big(\varphi_k(t,x,\xi)+\varphi_k(t,x,\rho\omega)\big)$ has, for each fixed (t,x), Jacobian factor comparable to $\delta(l)^2$. The estimate (3.19) now follows from Schur's Lemma and Lemma 3.4.

To conclude the proof of Theorem 3.1, we split $T^{l,\omega}$ into a finite number of pieces so that we may assume that

$$\big(T^{l,\omega}_{\nu,j,s,z}\big)^* T^{l,\omega}_{\nu',j',s',z'} = 0$$

unless $z = z'$ and $s = s$, and

$$T^{l,\omega}_{\nu',j',s',z'}\big(T^{l,\omega}_{\nu,j,s,z}\big)^* = 0$$

unless $\nu = \nu'$ and $j = j'$. We now consider an arbitrary finite truncation of the following sum to M elements

$$T^{l,\omega} = \sum_{\nu,j,s,z} T^{l,\omega}_{\nu,j,s,z}$$

The proof of the Cotlar-Stein Lemma yields the following,

$$\begin{aligned}\big\|\,T^{l,\omega}\,\big\|^{2N} \le C\,\delta(l)^{-1}\sum \big\|\,\big(T^{l,\omega}_{\nu_1,j_1,s_1,z_1}\big)^* T^{l,\omega}_{\nu_2,j_2,s_1,z_1}\,\big\|^{\frac12}\,\big\|\,T^{l,\omega}_{\nu_2,j_2,s_1z_1}\big(T^{l,\omega}_{\nu_2,j_2,s_2z_2}\big)^*\,\big\|^{\frac12}\\ \big\|\,\big(T^{l,\omega}_{\nu_2,j_2,s_2z_2}\big)^* T^{l,\omega}_{\nu_3,j_3,s_2z_2}\,\big\|^{\frac12}\cdots\big\|\,\big(T^{l,\omega}_{\nu_N,j_N,s_N,z_N}\big)^* T^{l,\omega}_{\nu_{N+1},j_{N+1},s_N,z_N}\,\big\|^{\frac12}\end{aligned}$$

and by estimates (3.17) and (3.19) this implies

$$\big\|\,T^{l,\omega}\,\big\|^{2N} \le M\,C^{2N}\,\delta(l)^{-\frac{3N}{2}}\,|\log\delta(l)|^N\,.$$

Letting $N \to \infty$ completes the proof of Theorem 3.1. $\square$

4. Nullform Estimates for the Wave Equation on Geodesically Concave Manifolds

In this section we work locally on a three-dimensional Riemannian manifold Ω with metric $\mathbf{g}$ and with smooth boundary $\partial\Omega$, such that Ω is strictly geodesically concave with respect to $\mathbf{g}$. The typical example is Ω the complement in $\mathbb{R}^3$ of a strictly convex open set with the Euclidean metric. By the Cauchy problem on Ω with Dirichlet condition we understand the following system

$$\begin{aligned}\partial_t^2 u(t,x) &= \Delta_g u(t,x) + F(t,x)\,,\\ u(t,x) &= 0 \quad \text{if} \quad x \in \partial\Omega\,,\\ u(0,x) &= u_0(x)\,,\\ \partial_t u(0,x) &= u_1(x)\,.\end{aligned}$$

We work in a local coordinate patch centered at the origin such that Ω is defined by $x_3 \ge 0$. For $k = 1, 2$ we set

$$H^k_D(\Omega) = \big\{ f \in H^k(\Omega) \,:\, f|_{\partial\Omega} = 0 \,\big\}\,,$$

where $H^k(\Omega)$ is the space of restrictions of elements of $H^k(\mathbb{R}^3)$.

Theorem 4.1. *Suppose that u and v satisfy the Cauchy problem on Ω with Dirichlet condition, with respective data*

$$u_0, v_0 \in H^2_D(\Omega)\,, \quad u_1, v_1 \in H^1_D(\Omega)\,, \quad F, G \in H^1([-c_0, c_0] \times \Omega)\,.$$

Suppose also that the data vanishes for $|x| \geq c_0$, where $c_0 > 0$ is a constant depending on Ω. Then the following holds, for any of the nullforms Q,

$$\begin{aligned}\big\|Q(du,dv)\big\|_{H^1([-c_0,c_0]\times\Omega)} \leq C\,\Big(\,\|u_0\|_{H^2_D(\Omega)} + \|u_1\|_{H^1_D(\Omega)} + \|F(s,\cdot)\|_{H^1([-c_0,c_0]\times\Omega)}\Big)\\ \times\,\Big(\,\|v_0\|_{H^2_D(\Omega)} + \|v_1\|_{H^1_D(\Omega)} + \|G(s,\cdot)\|_{H^1([-c_0,c_0]\times\Omega)}\Big)\,.\end{aligned}$$

Proof. For convenience, in this proof we refer to the discussion in [SmSo] regarding the parametrix for the Dirichlet problem; however, all of the results used are due to Melrose-Taylor [MT1-3] and Zworski [Z].

We may assume that Ω is a compact manifold, hence that the Dirichlet Laplacian $-\Delta$ is strictly positive on L^2. Using energy estimates and symmetry, we are reduced to the estimate where the left hand side is replaced by $\big\|Q(d\partial u, dv)\big\|_{L^2([-c_0,c_0]\times\Omega)}$, where ∂u is any space or time derivative of u. We next reduce matters to the following pair of estimates for the homogeneous problem,

$$\begin{aligned}\big\|Q(d\,\partial_x u,dv)\big\|_{L^2([-c_0,c_0]\times\Omega)} &\leq C\,\Big(\,\|u_0\|_{H^2_D(\Omega)} + \|u_1\|_{H^1_D(\Omega)}\Big)\Big(\,\|v_0\|_{H^2_D(\Omega)} + \|v_1\|_{H^1_D(\Omega)}\Big)\,,\\ \big\|Q(du,dv)\big\|_{L^2([-c_0,c_0]\times\Omega)} &\leq C\,\Big(\,\|u_0\|_{H^1_D(\Omega)} + \|u_1\|_{L^2(\Omega)}\Big)\Big(\,\|v_0\|_{H^2_D(\Omega)} + \|v_1\|_{H^1_D(\Omega)}\Big)\,.\end{aligned} \tag{4.0}$$

We first reduce the theorem to the case $G=0$. To this end, write

$$\begin{aligned}\int_0^t \frac{\sin\big((t-s)\sqrt{-\Delta}\,\big)}{\sqrt{-\Delta}}\,G(s,x)\,ds = -\cos\big(t\sqrt{-\Delta}\,\big)\Delta^{-1}\,G(0,x) + \Delta^{-1}\,G(t,x)\\ + \int_0^t \cos\big((t-s)\sqrt{-\Delta}\,\big)\Delta^{-1}\,\partial_s G(s,x)\,.\end{aligned} \tag{4.0a}$$

From the fact that $\partial_t G(t,x) \in L^2_t L^2_x$, and the fact that $G \in L^2_t L^6_x$, Sobolev embedding implies that

$$d\,\Delta^{-1}G(t,x) \in L^2_t L^\infty_x\,;$$

since $d\partial u(t,x) \in L^\infty_t L^2_x$, the contribution of this term to v is suitably bounded. The term in dv where $d = \partial_t$ acts on the upper limit in the time integral on the right hand side of $(4.0a)$ is similarly handled. Since $\Delta^{-1}\,G(0,x) \in H^2_D(\Omega)$, and $\Delta^{-1}\,\partial_s G(s,x) \in L^2_t H^2_D(\Omega)$, the theorem is now easily reduced to the case $G=0$.

Next consider $Q(d\partial_x u, dv)$. We apply the identity $(4.0a)$ to the inhomogeneity F. Since $dv \in L^\infty_t L^6_x$ and $d\,\partial_x \Delta^{-1} F(t,x) \in L^2_t L^6_x$, the estimate for $\|Q(d\,\partial_x u, dv)\|_{L^2}$ is easily reduced to the case $F=0$. To handle $Q(d\,\partial_t u, dv)$, we note that $\partial_t u$ solves the Cauchy problem with data in $H^1_D(\Omega) \times L^2(\Omega)$, with inhomogeneity in $L^2_t L^2_x$, and controlling $\|Q(d\,\partial_t u, dv)\|_{L^2}$ is reduced to the second estimate in (4.0).

Then, as discussed in [SmSo] immediately preceeding formulas (2.12) and (2.24) of that paper, for some c_0 as in the statement of the theorem, the solution u may be written, modulo smoothing operators acting on the data, as a finite sum of terms of the form

$$Tf(t,x) = \int e^{i\varphi^{\pm}(t,x,\xi)}\,a(t,x,\xi)\,\widehat{f}(\xi)\,d\xi\,,$$

where the phases are the solutions to the eikonal equation for some smooth extension of the metric $\mathbf{g}$ to an open neighborhood of the origin in $\mathbb{R}^3$, and the data $f \in H^2(\mathbb{R}^3)$ satisfies

$$\|f\|_{H^2(\mathbb{R}^3)} \le C\left(\|u_0\|_{H^2_D(\Omega)} + \|u_1\|_{H^1_D(\Omega)}\right).$$

The amplitude is one of two types. Either it satisfies the modified $S^0_{\frac{2}{3},\frac{1}{3}}$ estimates $(1.3a)$ of this paper, or it satisfies the following estimates:

$$\left| x_3^j \partial_{x_3}^k \langle \xi, \partial_\xi\rangle^N \partial^\beta_{t,x_1,x_2} \partial^\alpha_\xi a(t,x,\xi)\right| \le C_{j,k,N,\alpha,\beta}\left(1+|\xi|\right)^{\frac{2}{3}(k-j-|\alpha|)+\frac{1}{3}|\beta|}. \tag{4.1}$$

(We remark that in [SmSo] these estimates on the symbol were shown to hold for $N=0$; that the estimates hold for general N follows from the fact that these modified estimates are preserved under the equivalence of phase theorem of Hörmander as seen, for example, by the asymptotic formula for the transformed symbol.)

In either case, the operator $\partial_x T$ is an operator of the same type, with a symbol of one higher order, hence the estimates (4.0), and consequently Theorem 4.1, are reduced to verifying the following estimate

$$\left\|Q(dTf, dTg)\right\|_{L^2(\mathbb{R}^{1+3}_{t,x})} \le C\,\|f\|_{H^1(\mathbb{R}^3)}\|g\|_{H^2(\mathbb{R}^3)}. \tag{4.2}$$

We remark that in [SmSo], the Strichartz estimates were shown to hold for both symbol types:

$$\|Tf\|_{L^4_tL^4_x(\mathbb{R}^{1+3})} \le C\,\|f\|_{H^{\frac{1}{2}}(\mathbb{R}^3)}.$$

We first verify that the reductions of the second section of this paper hold for symbols satisfying the estimates (4.1). There are two places where the arguments need to be modified. The first is to verify that the estimate (4.2) holds if, in the formula for dT, the d acts on the symbol. Consider the term dTf, where the d hits the symbol satisfying (4.1). In this case, one obtains an operator Sf of the same form but with symbol of order $\frac{2}{3}$. The resulting contribution to the left hand side of (4.2) is controlled by noting that

$$\left\|(Sf)\,(dTg)\right\|_{L^2_{t,x}} \le \|Sf\|_{L^6_tL^3_x}\,\|dTg\|_{L^\infty_tL^6_x} \le C\|f\|_{H^1}\,\|g\|_{H^2},$$

where the last estimate for f follows by interpolating the following estimates

$$\|Sf\|_{L^4_tL^4_x} \le C\,\|f\|_{H^{\frac{7}{6}}(\mathbb{R}^3)},$$

$$\|Sf\|_{L^\infty_tL^2_x} \le C\,\|f\|_{H^{\frac{2}{3}}(\mathbb{R}^3)}.$$

Similarly one may bound

$$\left\|(dTf)\,(Sg)\right\|_{L^2_{t,x}} \le \|dTf\|_{L^\infty_tL^2_x}\,\|\,|D_x|^{\frac{5}{6}}Sg\|_{L^4_tL^4_x} \le C\,\|f\|_{H^1}\,\|g\|_{H^2}.$$

The other modification is to verify that the operator (2.6) has norm of order $2^{-\frac{k}{4}}$, in case the symbol $a_k(t,x,\xi)$ satisfies (4.1). This follows by expressing

$$Af(x) = \int_0^{x_n} 2^{\frac{2}{3}k}\left(1+2^{\frac{4}{3}k}r^2\right)^{-1} A_r f(x)\,dr$$

where A_r is the operator obtained by replacing $a_k(t,x,\xi)$ by the symbol

$$a_{k,r}(t,\overline{x},\xi) = 2^{-\frac{2}{3}k}\,\big(1+2^{\frac{4}{3}k}r^2\big)\,\partial_{x_n}a_k(t,\overline{x},r,\xi)\,, \qquad \overline{x}=(x_1,\,\dots\,,x_{n-1})$$

which satisfies, for each r, the estimates $(1.3a)$, with constants independent of r. One then has the bound

$$\|Af\|_{L^2(\mathbb{R}^3_x)} \le \sup_r \|A_r f\|_{L^2(\mathbb{R}^3_x)} \le C\,2^{-\frac{k}{4}}\,.$$

This procedure of "freezing the x_3 coefficient" will be used in subsequent steps.

We are thus reduced to establishing estimate (3.1). The above technique of freezing the x_3 coefficient reduces to the case that the symbol $a_k(t,x,\xi)$ in formula (2.3) satisfies the good estimates $(1.3a)$, and the symbol $a_j(t,x,\rho\omega)$ satisfies the estimates (4.1) above.

We next note that the proofs of Theorems 3.1 and 3.2 go through if $\widehat{g}(\rho)$ is supported in the region where $\rho\le 2^{\frac{3k}{4}}$. This follows since, in this case, we have $2^{\frac{2j}{3}}\le 2^{\frac{k}{2}}$, hence ∂_x loses at most $2^{\frac{k}{2}}$ against the symbol $a_j(t,x,\rho\omega)$. The only step in the proof that needs to be modified is to replace the right hand side of (3.8) by

$$C\,\Big(1+\delta(l)^2\,2^{\max(j,j')/3}\Big)^{-1}\,\|g_j\|_{L^2(\mathbb{R})}\,\|g_{j'}\|_{L^2(\mathbb{R})}\,,$$

to reflect the $(\frac{2}{3},\frac{2}{3})$ estimates on $a_j(t,x,\rho\omega)$.

We thus assume that $\widehat{g}(\rho)$ is supported in the region where $\rho\ge 2^{\frac{3k}{4}}$. Notice that $\rho\ge 2^{\frac{k}{2}}\,\delta(l)^{-1}$, since $\delta(l)\ge 2^{-\frac{k}{4}}$. Consequently $T^{l,\omega}(f,g)=T^{l,\omega}_1(f,g)$. We will show that

$$\|\,T^{l,\omega}(f,g)\|_{L^2(dx\,dt)} \le\; C\,\delta(l)^{-\frac{3}{4}}\,|\log\delta(l)|^{\frac{3}{2}}\;\|f\|_{L^2(\mathbb{R}^3)}\,\|g\|_{L^2(\mathbb{R})}\,. \tag{4.3}$$

We do this by setting

$$\tilde{T}^{l,\omega}_0(f,g)=\sum_j T^{l,\omega}(f,g_j)\;:\;2^{\frac{3k}{4}}\le 2^j\le 2^{\frac{2k}{3}}\,\delta(l)^{-1}\,,$$
$$\tilde{T}^{l,\omega}_1(f,g)=\sum_j T^{l,\omega}(f,g_j)\;:\;2^j> 2^{\frac{2k}{3}}\,\delta(l)^{-1}\,.$$

For the term $\tilde{T}^{l,\omega}_0(f,g)$, the index j runs over at most $|\log\delta(l)|$ terms. Thus, the bound (4.3) for this term results from the bound

$$\|\,T^{l,\omega}(f,g_j)\|_{L^2(dx\,dt)} \le\; C\,\delta(l)^{-\frac{3}{4}}\,|\log\delta(l)|^{\frac{1}{2}}\;\|f\|_{L^2(\mathbb{R}^3)}\,\|g_j\|_{L^2(\mathbb{R})}\,.$$

This estimate follows from the argument for $(\frac{1}{2},\frac{1}{2})$ symbols by freezing the x_3 coefficient in $a_j(t,x,\rho\omega)$.

To handle the term $\tilde{T}^{l,\omega}_1(f,g)$, we modify the argument of Theorem 3.1 by taking the partition of unity such that the symbol $a^{l,\omega}_{\nu,j,s,z}(t,x,\xi,\rho)$ is supported in the set

$$\Big|\,\frac{\xi}{|\xi|}-\nu\,\Big|\le c\,2^{-\frac{k}{3}}\,\delta(l)^{-1}\,,\qquad \big|\,\nabla_\xi\varphi_k(t,x,\omega)-z\,\big|\le 2\cdot 2^{-\frac{2k}{3}}\,,\qquad |\,t-s\,|\le 2\cdot 2^{-\frac{2k}{3}}\,\delta(l)^{-1}\,,$$

and modifying the spacing of the index points (ν, s, z) accordingly. With this modification, and using the modified $S_{\frac{2}{3},\frac{2}{3}}$ estimates for the symbol, estimates (3.17) and (3.19) are respectively replaced by

$$\| T^{l,\omega}_{\nu,j,s,z}\big(T^{l,\omega}_{\nu,j,s',z'}\big)^* \| \;\le\; \frac{C_N\,\delta(l)^{-2}}{\big(1+2^{\frac{2k}{3}}\,|\,z-z'\,|+2^{\frac{2k}{3}}\,\delta(l)\,|\,s-s'\,|\big)^N}$$

$$\| \big(T^{l,\omega}_{\nu,j,s,z}\big)^*\, T^{l,\omega}_{\nu',j',s,z} \| \;\le\; \frac{C_N\,\delta(l)^{-1}}{\big(1+2^{-\frac{2k}{3}}\,\delta(l)^2\,\big|2^j-2^{j'}\big|+2^{\frac{k}{3}}\,\delta(l)\,|\,\nu-\nu'\,|\big)^N}\,.$$

Since the indices now run over $2^j\,\delta(l)\ge 2^{\frac{2k}{3}}$, the rest of the proof of Theorem 3.1 follows. □

References

[BB] M. Beals and M. Bezard, *Low regularity local solutions for field equations*, Comm. Partial Differential Equations **21** (1996), 79–124.

[KM] S. Klainerman and M. Machedon, *Space-time estimates for null forms and the local existence theorem*, Comm. Pure. Appl. Math. **46** (1993), 1221–1268.

[MT1] R. Melrose and M. Taylor, *Near peak scattering and the corrected Kirchoff approximation for a convex obstacle*, Adv. Math. **55** (1985), 242–315.

[MT2] ———, *The radiation pattern of a diffractive wave near the shadow boundary*, Comm. Partial Diff. Equations **11** (1985), 599–672.

[MT3] ———, *Boundary problems for the wave equation with grazing and gliding rays*, manuscript.

[SSS] A. Seeger, C. Sogge, and E.M. Stein, *Regularity properties of Fourier integral operators*, Annals Math. **133** (1991), 231–251.

[Sm1] H. Smith, *A parametrix construction for wave equations with $C^{1,1}$ coefficients*, Annales de l'Institut Fourier **48** (1998).

[Sm2] ———, *Strichartz and nullform estimates for metrics of bounded curvature*, Preprint.

[SmSo] H. Smith and C. Sogge, *On the critical semilinear wave equation outside convex obstacles*, Jour. Amer. Math. Soc. **8** (1995), 879–916.

[So] C. Sogge, *On local existence for nonlinear wave equations satisfying variable coefficient null conditions*, Comm. PDE **18** (1993), 1795–1821.

[Z] M. Zworski, *High frequency scattering by a convex obstacle*, Duke Math. J. **61** (1990), 545–634.