Lp Bounds for Spectral Clusters for Lipschitz Metrics

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$a^{ij}(x) > 0$, $\rho(x) > 0$, Lipschitz

- Eigenbasis: $D_i\big(\, a^{ij} D_j \phi_j \big) = \lambda_j^2 \, \rho \, \phi_j \qquad (\, \lambda_j =$ frequency $)$

- Spectral Cluster, frequency λ :

$$u = \sum_{\lambda_j \in [\lambda, \lambda+1]} c_j \, \phi_j$$

- Goal: find sharp powers $\delta(p)$ such that

$$\frac{\|u\|_{L^p(M)}}{\|u\|_{L^2(M)}} \lesssim \lambda^{\delta(p)} \qquad (\, p \geq 2 \,)$$

Reduce problem to dispersive estimates:

- Localize $\widehat{u}(\xi)$ near ξ_1-axis
- Set $x_1 = t$, $x_2 = x$, factor

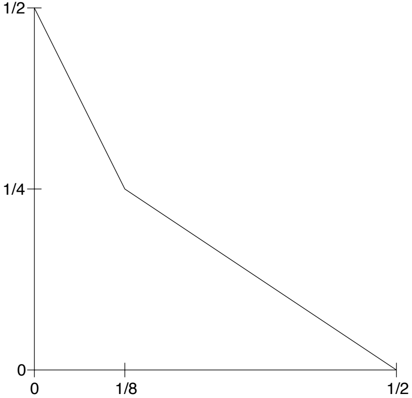
$$-D_i\, a^{ij} D_j + \lambda^2 \rho = a^{00} \big(\partial_t + iP_\lambda(t,x,D_x)\big)\big(\partial_t - i\tilde{P}_\lambda(t,x,D_x)\big)$$

- Prove for $\big(\partial_t + iP_\lambda(t,x,D_x)\big) u = 0$

$$\|u\|_{L^p} \lesssim \lambda^{\delta(p)} \|u_0\|_{L^2}$$

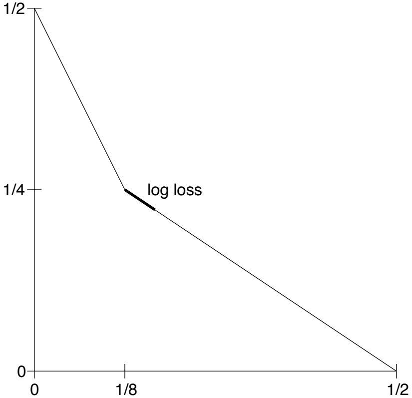
Estimates no better than domains with boundary.

Metric $d_{x_2}^2 + (1 + |x_2|)\, d_{x_1}^2 \;\approx\;$ interior of disc



Koch-S.-Tataru [2010]

Best possible estimates in dimension $n = 2$, for $8 < p \leq \infty$, and for $p = 8$ with loss of $(\log \lambda)^{\alpha}$.



For $p = 6$, hold by short-time $|t| \leq \lambda^{-1/3}$ parametrix.

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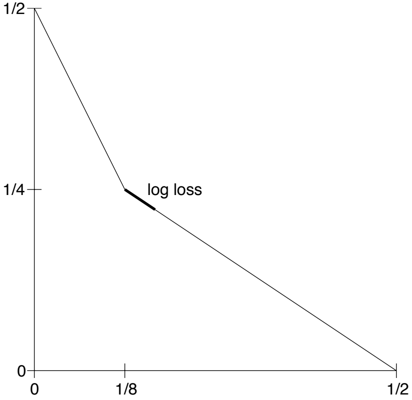

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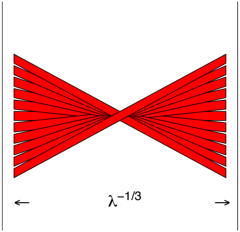
Short time parametrices alone can't yield sharp results

A single angle-1 bush (conically localized zonal eigenfunction) saturates L^p estimates, $p \geq 8$



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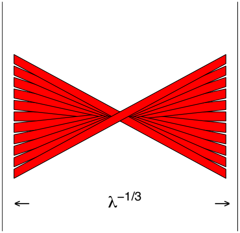
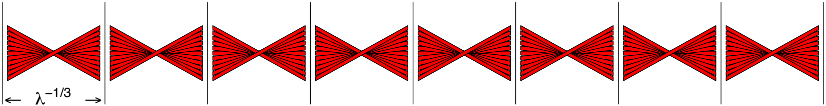




$\lambda^{1/3}$ terms $\Rightarrow$ loss of $\lambda^{1/3p}$ in estimates

Need control energy flow for $|t| > \lambda^{-1/3}$

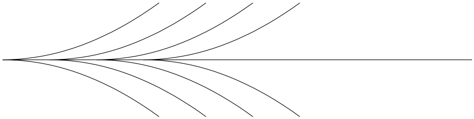
Problem: bi-characteristic flow not well-posed

$$\dot{x} = p_\xi(t,x,\xi) \in Lip_x\,, \qquad \dot{\xi} = p_x(t,x,\xi) \in L^\infty_x$$

All that you can control:

$$|\ddot{x}| \lesssim 1\,, \qquad |\dot{\xi}| \lesssim \lambda$$

Metric $d_{x_2}^2 + (1-|x_2|)d_{x_1}^2 \;\Rightarrow$ bifurcation:

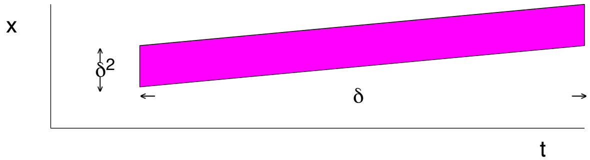


Stable regions of phase space for time δ:

$$|x - x_0| \le \delta^2 , \qquad |\xi - \xi_0| \le \lambda\delta$$

Integral curves through (x, ξ) satisfy

$$|x(t) - v_0 t - x_0| \lesssim \delta^2 , \qquad |\xi(t) - \xi_0| \lesssim \lambda\delta , \quad |t| \le \delta$$



Uncertainty principle: $\delta \ge \lambda^{-1/3}$

Heuristics behind energy control, $|\ddot{x}| \lesssim 1$, $|\dot{\xi}| \lesssim \lambda$

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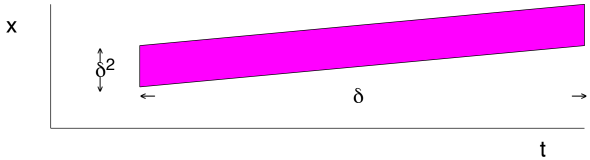

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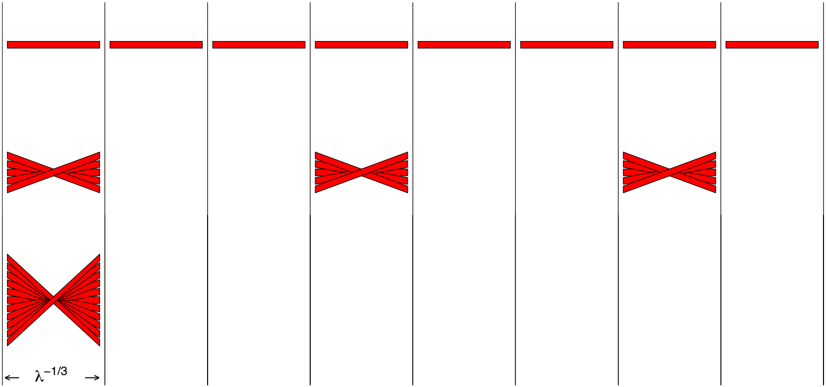
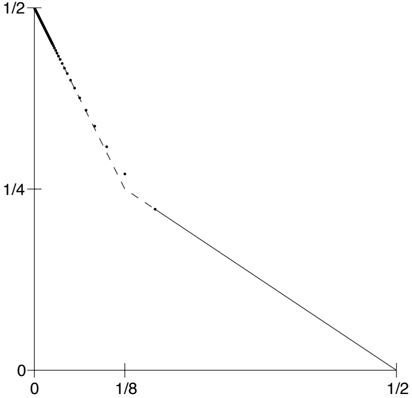
Angle θ bush can reoccur only after time θ



First use of energy flow arguments...

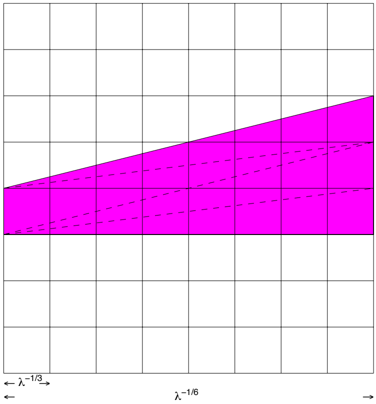
Koch-S.-Tataru [2008]

For $p = 8, 10, 12, \ldots$, loss of $2^{(6-p)/2}/3p$



Tube overlap count $\Rightarrow \ell^q$(cubes) bounds on energy

Induction: log loss L^p on δ^2 cubes $\Rightarrow$ log loss L^{p+2} on δ cubes.

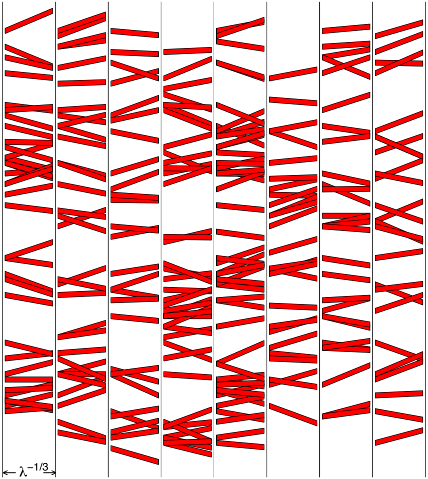


$p = 8$: log-loss estimates on slabs $\Delta t \le \lambda^{-1/6}$

New proof: expand u in short-time tube solutions

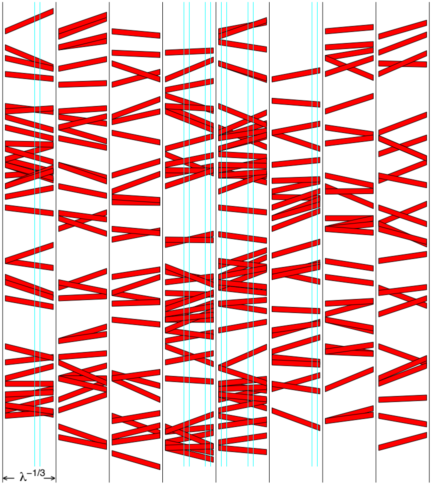
Set $\|u_0\|_2 = 1$, expand u in tube frame each $\lambda^{-1/3}$ time slab.

At cost of $\log \lambda$, consider $u_a =$ with tubes of amplitude $\approx a$.



Identify regions with overlap 2^m

2^m-bushes remains overlapped for time $\leq 2^{-m}\lambda^{-\frac{1}{3}}$



Two key propositions: amplitude a tubes, overlap 2^m

Proposition 1: bush counting

There are at most $\approx \lambda^{1/3}\, 2^{-3m} a^{-4}$ intervals that contain a 2^m-bush

Energy-1 bush has $2^m\, a^2 = 1$; at most $\lambda^{1/3}\, 2^{-m}$ such bushes.

Proposition 2: local L^8 bounds

On each interval, where $A_{a,m} = 2^m$-overlap region,

$$\|u_a\|_{L^8(I \cap A_{a,m})} \lesssim \lambda^{5/24}\, 2^{3m/8}\, a^{1/2}\,.$$

Sum over I, $\|u_a\|_{L^8(A_{a,m})} \lesssim \lambda^{1/4}$, log-loss in sum over m.

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Local L^8: geometry essentially flat each $\lambda^{-1/3}$ interval

Key ingredients:

- Bi-linear estimates handle large angle interactions.
- Strichartz estimates handle small angle interactions.

  Tube / wave packet representation of solutions well-adapted to proving both bilinear and Strichartz in low dimensions.

- On 2^m-overlap region $A_{a,m}$ have L^∞ bounds.

  Interpolate with L^4 and L^6 to get L^8.

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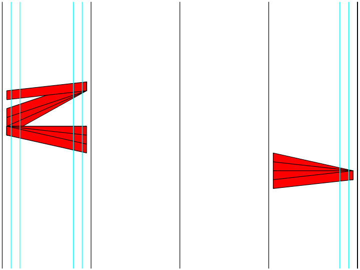
Bush-counting: energy flow for Lipschitz metrics

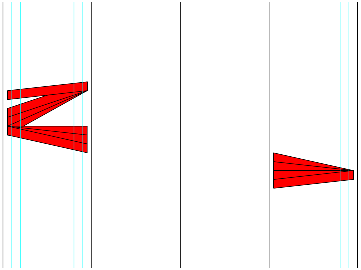
Key estimate: bound energy coupling between 2^m-bushes at distinct times.



$S(t, t') =$ evolution operator for $\partial_t + iP(t, x, D_x)$

Let P_j be projection onto 2^m-bush at time t_j:

$$\|P_1\, S(t_1, t_0)\, P_0\|_{L^2 \to L^2} \lesssim 2^{-m} \lambda^{-1/3} |t_1 - t_0|^{-1} + 2^{-m} \lambda^{1/3} |t_1 - t_0|$$



Higher dimensions: sharp bounds on smaller range

Main problem: wrong decay for bush-interaction $P_1 S(t_1, t_0) P_0$.

Gives sharp estimates for large p:

[Koch-S.-Tataru] Lipschitz metrics, dimension n,

$$\|\Pi_{[\lambda,\lambda+1]} u\|_p \lesssim \lambda^{\frac{n-1}{2} - \frac{n}{p}}, \qquad \frac{4n+2}{n-1} < p \leq \infty\,.$$

Short time parametrix gives sharp estimates for small p:

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$$\|\Pi_{[\lambda,\lambda+1]} u\|_p \lesssim \lambda^{\frac{2}{3}(n-1)(\frac{1}{2} - \frac{1}{p})}, \qquad 2 \leq p \leq \frac{2n+2}{n-1}\,.$$

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Dimension $n = 3$: critical estimate is $\lambda^{2/5}$ for $p = 5$