Concentration of Eigenfunctions in Rough Media

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M = compact manifold with volume form

- Elliptic form $a^{ij}(x)$, weight function $\rho(x)$
  Eigenbasis: $D(a\,D\phi_j) = \lambda_j^2\,\rho\,\phi_j \quad (\,\lambda_j = \text{frequency}\,)$
- Spectral Cluster, frequency λ :

$$f = \sum_{\lambda_j \in [\lambda, \lambda+1]} c_j\,\phi_j$$

- Goal: find sharp powers $\delta(p)$ such that

$$\frac{\|f\|_{L^p(M)}}{\|f\|_{L^2(M)}} \lesssim \lambda^{\delta(p)} \qquad (\,p \geq 2\,)$$

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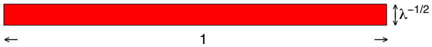
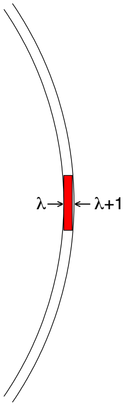
Saturating examples on $\mathbb{R}^n$: $1 \times \lambda^{-1/2}$ tube

$$f(x) = e^{i\lambda x_1}\psi(x_1, \lambda^{1/2}x')$$

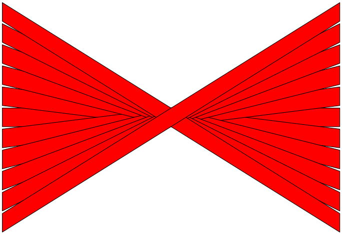


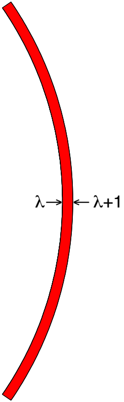


Lower bound

$$\delta(p) \geq \frac{(n-1)}{2}\left(\frac{1}{2} - \frac{1}{p}\right)$$

Saturating examples on $\mathbb{R}^n$: angle-1 bush





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$$\delta(p) \geq n(\tfrac{1}{2} - \tfrac{1}{p}) - \tfrac{1}{2}$$

Theorem: Sogge [1988]

If $a^{ij}(x)\,,\rho(x)\ \in C^\infty(M)$, then $\|f\|_{L^p(M)} \lesssim \lambda^{\delta(p)}\|f\|_{L^2(M)}$,

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Same results hold for $a^{ij}(x)\,,\rho(x)\ \in C^{1,1}(M)$.

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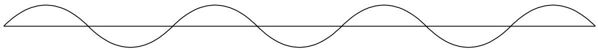
Singular bicharacteristic flow:



"Tube"-eigenfunction $f(x)$ exponentially localized to $|x'| \leq \lambda^{-\frac{2}{2+s}}$

Lower bound for C^s coefficients

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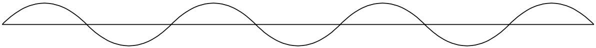

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- C^2-scale $R = \lambda^{-\frac{2-s}{2+s}}$: a spectral cluster for C^s metric rescaled by R behaves like spectral cluster for C^2 metric.
- Sogge's estimates hold on sets of size R : sum over pieces get sharp results for $p = \frac{2(n+1)}{n-1}$ and $p = \infty$.

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For $s < 1$, exponentially localized eigenfunctions

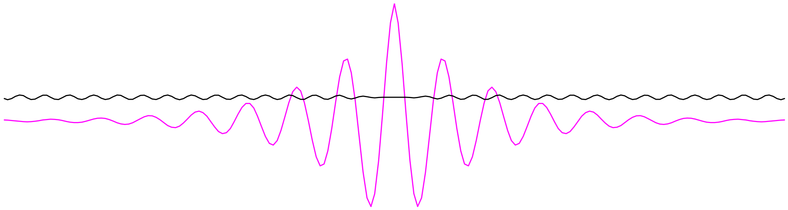
Colombini-Spagnolo [1989]

There exist $\rho_\epsilon(t) = 1 + \epsilon\, q_1(t) + \epsilon^2 q_2(t)$ with eigenfunction

$$w_\epsilon''(t) + \rho_\epsilon(t) w_\epsilon(t) = 0$$

with

$$|w_\epsilon(t)| \lesssim e^{-\epsilon|t|}$$



For $0 \le s < 1$, exponentially localized eigenfunctions

Castro-Zuazua [2002]

Take $\epsilon = \lambda^{-s}$, rescale, then $\rho_{\lambda^{-s}}(\lambda x) \in C^s(\mathbb{R})$, eigenfunction

$$|w(x)| \lesssim e^{\lambda^{1-s}|x|}$$

Koch-Smith-Tataru [2006]

Constructed $C^s(\mathbb{R}^n)$ functions $a^{ij}(x)$, $\rho(x)$ with:

- Radial (bush) eigenfunctions localized to $|x| \le \lambda^{s-1}$
- Tube eigenfunctions localized to $|x'| \lesssim \lambda^{-\frac{2}{2+s}}$, $|x_1| \lesssim \lambda^{s-1}$

$$\frac{\|f\|_{L^p}}{\|f\|_{L^2}} \gtrsim \lambda^{n(\frac{1}{2}-\frac{1}{p})-\frac{s}{2}}, \qquad \frac{\|f\|_{L^p}}{\|f\|_{L^2}} \gtrsim \lambda^{\left(\frac{2}{2+s}(n-1)+1-s\right)(\frac{1}{2}-\frac{1}{p})}$$

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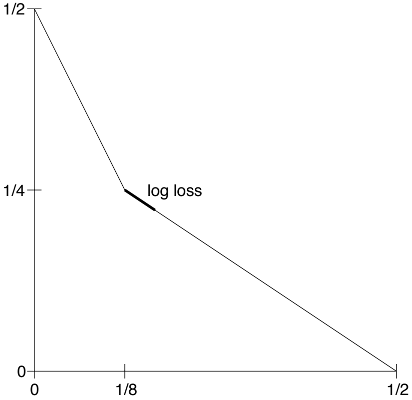
Koch-Smith-Tataru [2012]

For a^{ij}, $\rho \in$ Lipschitz, best possible estimates in dimension $n = 2$, up to loss of $(\log \lambda)^{\alpha}$ for $6 < p \leq 8$.



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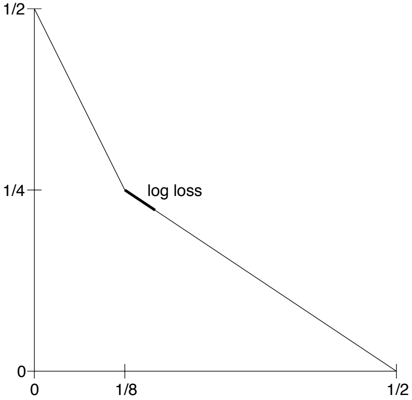

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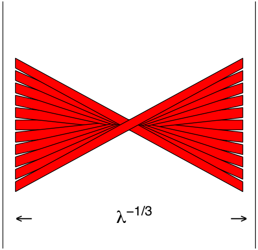
A single angle-1 bush saturates L^p estimates over M for $p \geq 8$.



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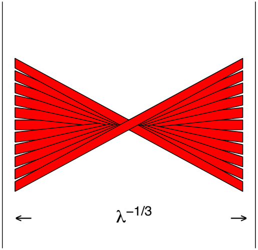
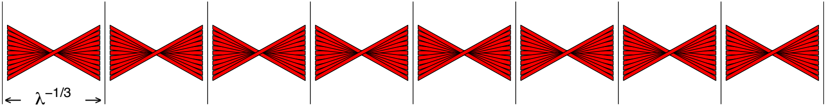




$\lambda^{1/3}$ terms $\Rightarrow$ loss of $\lambda^{1/3p}$ in estimates

Need control energy flow over scales $\gg \lambda^{-1/3}$

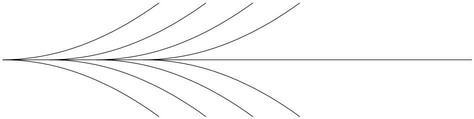
Problem: bi-characteristic flow not well-posed

$$\dot{x} = p_\xi(t,x,\xi) \in Lip_x S^0_\xi\,, \qquad \dot{\xi} = p_x(t,x,\xi) \in L^\infty_x S^1_\xi$$

All that you can control:

$$|\ddot{x}| \lesssim 1\,, \qquad |\dot{\xi}| \lesssim \lambda$$

Metric $d^2_{x_2} + (1-|x_2|)d^2_{x_1} \;\Rightarrow\;$ bifurcation:



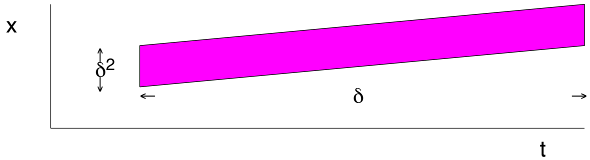
Heuristics behind energy control, $|\ddot{x}| \lesssim 1\,,\ |\dot{\xi}| \lesssim \lambda$

Stable regions of phase space for time δ:

$$|x - x_0| \le \delta^2\,, \qquad |\xi - \xi_0| \le \lambda\delta$$

Integral curves through (x, ξ) satisfy

$$|x(t) - v_0 t - x_0| \lesssim \delta^2\,, \qquad |\xi(t) - \xi_0| \lesssim \lambda\delta\,, \quad |t| \le \delta$$



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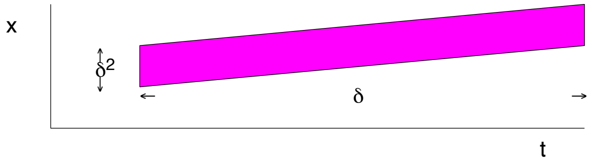

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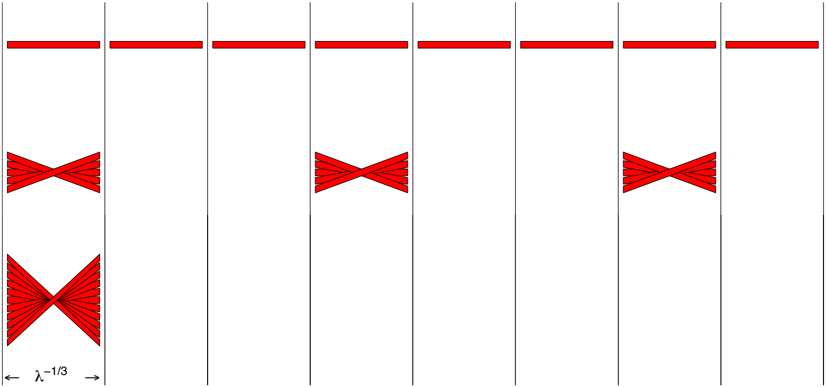
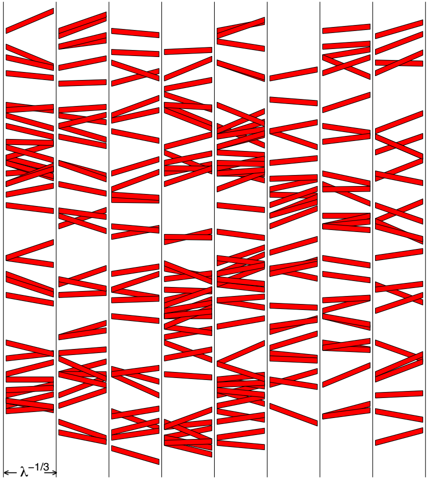
Angle θ bush can reoccur only after time θ



New proof: expand u in short-time tube solutions

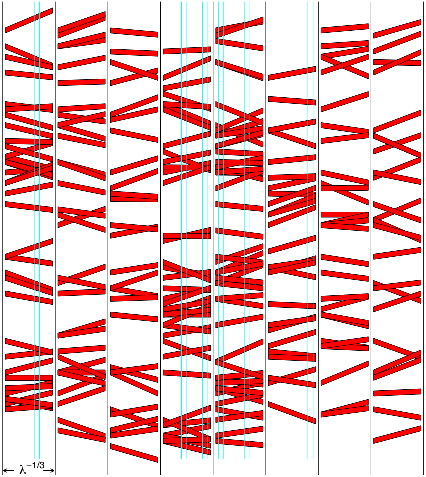
Set $\|u_0\|_2 = 1$, expand u in tube frame each $\lambda^{-1/3}$ time slab.

At cost of $\log \lambda$, consider $u_a =$ with tubes of amplitude $\approx a$.



Identify regions with overlap 2^m

2^m-bushes remains overlapped for time $\leq 2^{-m}\lambda^{-\frac{1}{3}}$



Two key propositions: amplitude a tubes, overlap 2^m

Proposition 1: bush counting

There are at most $\approx \lambda^{1/3}\, 2^{-3m} a^{-4}$ intervals that contain a 2^m-bush

Energy-1 bush has $2^m\, a^2 = 1$; at most $\lambda^{1/3}\, 2^{-m}$ such bushes.

Proposition 2: local L^8 bounds

On each interval, where $A_{a,m} = 2^m$-overlap region,

$$\|u_a\|_{L^8(I\cap A_{a,m})} \lesssim \lambda^{5/24}\, 2^{3m/8}\, a^{1/2}\,.$$

Sum over I, $\|u_a\|_{L^8(A_{a,m})} \lesssim \lambda^{1/4}$, log-loss in sum over m.

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Local L^8: geometry essentially flat each $\lambda^{-1/3}$ interval

Key ingredients:

- Bi-linear estimates handle large angle interactions.
- Strichartz estimates handle small angle interactions.

  Tube / wave packet representation of solutions well-adapted to proving both bilinear and Strichartz in low dimensions.

- On 2^m-overlap region $A_{a,m}$ have L^∞ bounds.

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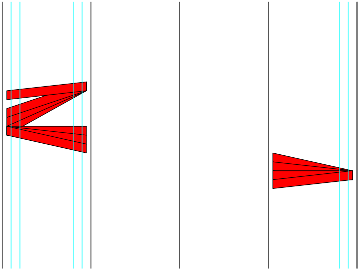
Bush-counting: energy flow for Lipschitz metrics

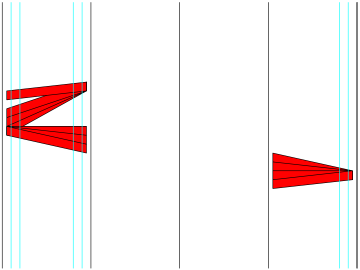
Key estimate: bound energy coupling between 2^m-bushes at distinct times.



$S(t,t') =$ evolution operator for $\partial_t + iP(t,x,D_x)$

Let P_j be projection onto 2^m-bush at time t_j:

$$\|P_1\, S(t_1,t_0)\, P_0\|_{L^2\to L^2} \lesssim 2^{-m}\lambda^{-1/3}|t_1-t_0|^{-1} + 2^{-m}\lambda^{1/3}|t_1-t_0|$$



Higher dimensions: sharp bounds on smaller range

Problem: wrong decay for bush-interaction $P_1 S(t_1, t_0) P_0$.

Gives sharp estimates for large p:

[Koch-S.-Tataru] Lipschitz metrics, dimension n,

$$\|\Pi_{[\lambda,\lambda+1]} u\|_p \lesssim \lambda^{n(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}} \|u\|_2 \,, \qquad \frac{6n-2}{n-1} < p \leq \infty \,.$$

Short time parametrix gives sharp estimates for small p:

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Dimension $n = 3$: critical estimate is $\lambda^{2/5}$ for $p = 5$