## An Introduction to Curvelets

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Curvelets

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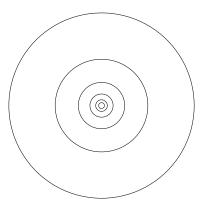
# A curvelet frame {φ<sub>γ</sub>} is a wave packet frame on L<sup>2</sup>(R<sup>2</sup>) based on second dyadic decomposition.

$$f(x) = \sum_{\gamma} c_{\gamma} \varphi_{\gamma}(x)$$
$$c_{\gamma} = \int f(x) \overline{\varphi_{\gamma}(x)} dx$$

Curvelets and the Second Dyadic Decomposition

#### **Dyadic Decomposition**

Frequency shells:  $2^k < |\xi| < 2^{k+1}$ 

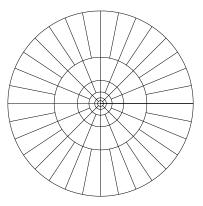


Curvelets and the Second Dyadic Decomposition

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#### Second Dyadic Decomposition

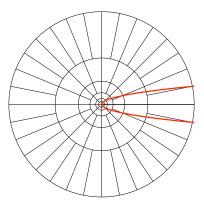
Angular Sectors:  $\angle(\omega,\xi) \leq 2^{-k/2}$ 



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#### Second Dyadic Decomposition

Parabolic scaling:  $\Delta \xi_2 \sim \sqrt{\xi_1}$ 



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#### Second Dyadic Decomposition

Associated partition of unity:

$$1 = \hat{\psi}_0(\xi)^2 + \sum_{k=0}^{\infty} \sum_{\omega=1}^{2^{k/2}} \hat{\psi}_{\omega,k}(\xi)^2$$

$$\operatorname{supp}(\hat{\psi}_{\omega,k}) \subset \left\{ \xi : |\xi| \approx 2^k, \ \left| \omega - \frac{\xi}{|\xi|} \right| \lesssim 2^{-k/2} \right\}$$

Second dyadic decomposition of *f*:

$$1 = \hat{\psi}_0(\xi)^2 \, \hat{f}(\xi) + \sum_{k=0}^{\infty} \sum_{\omega=1}^{2^{k/2}} \hat{\psi}_{\omega,k}(\xi)^2 \, \hat{f}(\xi)$$

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## Second Dyadic Decomposition

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# Final step: expand $\hat{\psi}_{\omega,k}(\xi) \hat{f}(\xi)$ in Fourier series

If supp $(g(\xi)) \subset L_1 \times L_2$  rectangle:

$$g(\xi) = (L_1 L_2)^{-1/2} \sum_{p,q} c_{p,q} e^{-ip\xi_1 - iq\xi_2}$$
$$c_{p,q} = (L_1 L_2)^{-1/2} \int g(\xi) e^{ip\xi_1 + iq\xi_2}$$

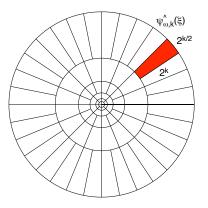
Points (p, q) belong to dilated lattice:

$$p,q\in rac{2\pi}{L_1}\mathbb{Z} imes rac{2\pi}{L_2}\mathbb{Z}$$

Curvelets and the Second Dyadic Decomposition

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 $\hat{\psi}_{\omega,k}(\xi)\hat{f}(\xi)$  supported in rotated  $2^k \times 2^{k/2}$  rectangle



Points (p, q) belong to rotated  $2^{-k} \times 2^{-k/2}$  lattice  $\Xi_{\omega,k}$ 

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## Let $(p, q) = x \in \Xi_{\omega, k}$

$$c_{x,\omega,k} = 2^{-3k/4} \int e^{i\langle x,\xi 
angle} \hat{\psi}_{\omega,k}(\xi) \hat{f}(\xi) d\xi$$

$$\hat{\psi}_{\omega,k}(\xi)\,\hat{f}(\xi)=\mathsf{2}^{-3k/4}\sum_{x\in\Xi_{\omega,k}}c_{\!x,\omega,k}\,e^{-i\langle x,\xi
angle}$$

Reconstruction is periodic, so localize:

$$\hat{\psi}_{\omega,k}(\xi)^2 \, \hat{f}(\xi) = 2^{-3k/4} \sum_{x \in \Xi_{\omega,k}} c_{x,\omega,k} \, e^{-i\langle x,\xi \rangle} \hat{\psi}_{\omega,k}(\xi)$$

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#### Sum over $\omega$ then k to recover f

$$\hat{f}(\xi) = \sum_{k} \sum_{\omega} \sum_{x \in \Xi_{\omega,k}} c_{x,\omega,k} \, 2^{-3k/4} e^{-i\langle x,\xi \rangle} \hat{\psi}_{\omega,k}(\xi)$$

$$c_{\mathbf{x},\omega,\mathbf{k}} = \int 2^{-3k/4} e^{i\langle \mathbf{x},\xi\rangle} \hat{\psi}_{\omega,\mathbf{k}}(\xi) \hat{f}(\xi) d\xi$$

Take inverse Fourier transform:

$$f(y) = \sum_{k} \sum_{\omega} \sum_{x \in \Xi_{\omega,k}} c_{x,\omega,k} 2^{-3k/4} \psi_{\omega,k}(y-x)$$

$$c_{x,\omega,k} = \int \overline{2^{-3k/4}\psi_{\omega,k}(y-x)} f(y) \, dy$$

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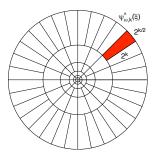
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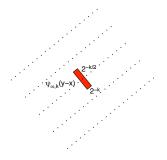
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 $\begin{array}{c} \begin{array}{c} \text{The Curvelet Frame} \\ \text{Applications of Curvelets} \end{array} \quad \text{Curvelets and the Second Dyadic Decomposition} \end{array}$ 

Frequency support:

#### Spatial support:





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#### **Plancherel identity**

$$\sum_{\boldsymbol{x}\in\Xi_{\omega,k}}|\boldsymbol{c}_{\boldsymbol{x},\omega,k}|^2=\int\hat{\psi}_{\omega,k}(\xi)^2|\hat{f}(\xi)|^2\,d\xi$$

Sum over  $\omega$  and k:

$$\sum_{\gamma} |c_{\gamma}|^2 = \int |\hat{f}(\xi)|^2 d\xi = \int |f(y)|^2 dy$$

*Frame, not basis:*  $\psi_{\omega,k}(y-x)$  *not orthogonal, but almost.* 

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#### Motivation for SDD: homogeneous phase functions

Consider  $\Phi(\xi)$  smooth for  $\xi \neq 0$ , homogeneous degree 1:

$$\Phi(s\xi) = s\Phi(\xi), \quad s > 0$$

*Example*:  $\Phi(\xi) = |\xi|$ 

Claim: on  $supp(\hat{\psi}_{\omega,k})$ ,  $\Phi(\xi) = \nabla \Phi(\omega) \cdot \xi + r(\xi)$ , where

$$|r(\xi)| \lesssim 1$$
  
 $|
abla_{\omega}^m 
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Application to wave flow Images with jumps along curves

## Example: $|\xi| = \omega \cdot \xi + r(\xi)$

Solution to half-wave Cauchy problem:

$$\left(\partial_t + i\sqrt{-\Delta_y}\right)u(t,y) = 0, \quad u(0,y) = \varphi_\gamma(y)$$

$$egin{aligned} & u(t, m{y}) = \int m{e}^{i \langle m{y}, \xi 
angle - it | \xi |} \, \hat{arphi}_{\gamma}(\xi) \, m{d}\xi \ &= \int m{e}^{i \langle m{y} - t \omega, \xi 
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Initial data:  $u(0, y) = \sum_{\gamma} c_{\gamma} \varphi_{\gamma}(y)$ , then

$$u(t,y) \approx \sum_{\gamma} \varphi_{\gamma}(y-t\omega)$$

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#### **Evolution of waves**

A wavefront consisting of a few curvelets:

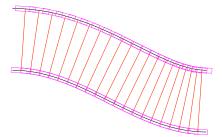


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#### Evolution of waves

First approximation to wave flow:

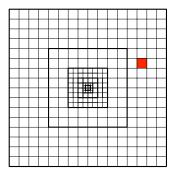


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## Córdoba-Fefferman Decomposition: $2^{k/2} \times 2^{k/2}$ cubes

Alternative phase-space decomposition being explored to compute wave propagators: (Demanét-Ying)

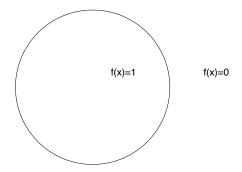


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## C. Fefferman [1973]

Decompose support function of unit disc:



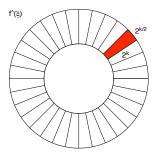
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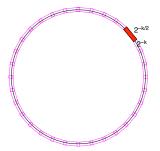
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## C. Fefferman [1973]

Frequency sectors:







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#### Candés-Donoho (2003)

Image with sharp jump along smooth curve:

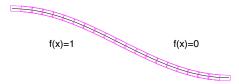


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#### Candés-Donoho (2003)

 $2^{k/2}$  dominant terms in curvelet expansion at frequency  $2^k$ :



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## Candés-Donoho (2003)

Approximation rate is optimal:

• Choose *n* largest coefficients  $c_{\gamma}$  in  $f = \sum_{\gamma} c_{\gamma} \varphi_{\gamma}$ 

$$\|f - f_n\|_{L^2}^2 \lesssim n^{-2} \log(n)^3$$

- No frame can do better for jumps along  $C^2$  curves.
- Wavelet expansion:  $||f f_n||_{L^2}^2 \lesssim n^{-1}$

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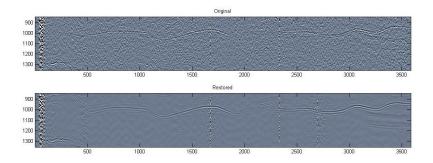
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## Denoising Images with Curvelets



Hart F. Smith An Introduction to Curvelets

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