Energy of flows on $\mathbf{Z}^2$ percolation clusters

Christopher Hoffman[1,2]

Abstract

We show that if $p > p_c(\mathbf{Z}^2)$, then the unique infinite percolation cluster supports a nonzero flow f with finite q energy for all $q > 2$. This extends the work of Grimmett, Kesten, and Zhang and Levin and Peres in dimensions $d \geq 3$. As an application of our techniques we exhibit a graph that has transient percolation clusters, but does not admit exponential intersection tails. This answers a question asked by Benjamini, Pemantle, and Peres.

Keywords : percolation, energy, electrical networks, exponential intersection tails.

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[1]Department of Mathematics, University of Maryland, College Park, MD, 20742

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1 Introduction

Consider Bernoulli bond percolation on $\mathbf{Z}^d$ with parameter p. Recall that this is the independent process on $\mathbf{Z}^d$ which retains an edge with probability p and deletes an edge with probability $1-p$. For all $d > 1$, there exists a critical parameter $p_c(\mathbf{Z}^d) < 1$ such that if $p < p_c$, then a.s. there is no connected component with infinitely many edges. If $p > p_c$, then a.s. there is a unique connected component with infinitely many edges [1]. This component is called the unique infinite cluster. Kesten proved that $p_c(\mathbf{Z}^2) = 1/2$ [9]. The phase where $p > p_c$ is called supercritical Bernoulli percolation. We consider only the supercritical case.

Let $G = G(E, V)$ be the graph with vertex set V and edge set E. Consider each undirected edge on G as two directed edges, one in each direction. Let vw be the directed edge from v to w. A **flow** f on G with source v_0 is an nonnegative edge function such that the net flow out of any vertex $v \neq v_0$ is zero: $\sum_w f(v_0 w) - \sum_w f(w v_0) = 0$. The **strength** of a flow f from the origin is the amount flowing from 0: $\sum_w f(0w) - \sum_w f(w0)$. A **multisource flow** f on G is the sum (or integral) of flows on G. The q **energy** of a flow (or multisource flow) f on G is

$$\mathcal{E}_q(f) = \sum_{e \in E} f(e)^q.$$

The **energy** of a flow is the 2 energy of the flow. Note that if a multisource flow on G has finite q energy then it is the sum (or integral) of flows with finite q energy.

Grimmett, Kesten and Zhang proved that if $d \geq 3$ and $p > p_c(\mathbf{Z}^d)$, then simple random walk on the infinite percolation cluster, $\mathcal{C}_\infty(\mathbf{Z}^d, p)$, is a.s. transient [6]. They proved this result by constructing a flow on the percolation cluster with finite energy, which is equivalent to transience of simple random walk on the cluster.

Grimmett, Kesten, and Zhang showed that there exists a tree in $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ where the branches bifurcate at fairly regular intervals. This tree gives rise to a flow in a natural way. It is then easy to bound the energy of this flow. Benjamini, Pemantle and Peres [3] gave an alternative proof of Grimmett, Kesten and Zhang's result. They constructed "unpredictable" processes on $\mathbf{Z}$. They used them to create a measure on the collection of paths in $\mathbf{Z}^d$ which emanate from 0. For $d \geq 3$ the measure μ they created has **exponential intersection tails**. That is, there exists C and a $\theta < 1$ such that for all n

$$\mu \times \mu\Big\{(\varphi, \psi) : |\varphi \cap \psi| \geq n\Big\} \leq C\theta^n,$$

where $|\varphi \cap \psi|$ is the number of edges in the intersection of φ and ψ. We say that a graph G **admits exponential intersection tails** if there exists a measure μ on paths on G with exponential intersection tails. Then they proved that any graph that admits exponential intersection tail has transient percolation clusters for some $p < 1$. They asked whether the converse is true. We show that it is not.

Levin and Peres adapted the approach of Benjamini et al. to show that for $d \geq 3$ these flows have finite q energy a.s. for $q > d/(d-1)$ [10]. This result is optimal since $\mathbf{Z}^d$ also supports flows of finite q energy if and only if $q > d/(d-1)$ [11]. In this paper we will use the method of Grimmett, Kesten, and Zhang to extend Levin and Peres's result to $d = 2$.

Theorem 1.1 *For every $p > p_c(\mathbf{Z}^2)$ and $q > 2$ there exists a flow on $\mathcal{C}_\infty(\mathbf{Z}^2, p)$ with finite q energy a.s.*

Levin and Peres also proved a result using a generalization of the energy of a flow. For any $d \geq 2$ and $\alpha > 0$, let

$$H_{d,\alpha}(u) = u^{d/(d-1)}/[\log(1 + u^{-1})]^\alpha$$

for $u > 0$ and $H_{d,\alpha}(0) = 0$. Levin and Peres proved that if $d \geq 3$ then $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ supports a flow of finite $H_{d,\alpha}$ energy for any $\alpha > 2$. (That is $\sum_e H_{d,\alpha}(f(e)) < \infty$.) Hoffman and Mossel sharpened this result by showing that $\mathcal{C}_\infty(\mathbf{Z}^d, p)$ supports a flow of finite $H_{d,\alpha}$ energy for $d \geq 3$ and $\alpha > 1$ [8]. This last result is optimal because there exists a flow of finite $H_{d,\alpha}$ energy on $\mathbf{Z}^d$ if and only if $\alpha > 1$ [12]. We will extend this result of Levin and Peres to $\mathbf{Z}^2$.

Theorem 1.2 *For every $p > 1/2$ and $\alpha > 2$ there exists a flow on $\mathcal{C}_\infty(\mathbf{Z}^2, p)$ with finite $H_{2,\alpha}$ energy a.s.*

We are unable to prove a version of Hoffman and Mossel's results for $d = 2$, but we conjecture that it is true.

Conjecture 1.1 *For every $p > 1/2$ and $\alpha > 1$ there exists a flow on $\mathcal{C}_\infty(\mathbf{Z}^2, p)$ with finite $H_{2,\alpha}$ energy a.s.*

This would require a new approach as the Grimmett et al. method cannot be extended to reach this conclusion and the Benjamini et al. approach says nothing about graphs, like $\mathbf{Z}^2$, that do not admit exponential intersection tails.

Häggström and Mossel used Benjamini, Pemantle and Peres's approach to extend the Grimmett, Kesten, and Zhang result in another way. They defined two classes of subgraphs of $\mathbf{Z}^3$ and showed these subgraphs admit exponential intersection tails. Thus simple random walk on the infinite percolation cluster of those subgraphs is transient [7]. On certain subgraphs of $\mathbf{Z}^3$ where Häggström and Mossel showed that simple random walk on the infinite percolation cluster is transient it is clear that the Grimmett, Kesten, and Zhang approach will not work. We give an example of a graph where the Grimmett et al. approach works but the Benjamini et al. approach does not. More specifically we prove the following.

Theorem 1.3 *There exists a graph G which has transient percolation clusters but does not admit exponential intersection tails.*

To do this we construct a graph G which is the direct product of a tree and $\mathbf{Z}$. The subset of G which projects onto a branch of the tree is a subgraph of $\mathbf{Z}^2$. On the percolation cluster restricted to that subset we construct a flow of finite $H_{2,3}$ energy. Integrating these flows over all branches gives a multisoure flow of finite energy on the percolation cluster of G. Thus simple random walk on the infinite percolation cluster of G is transient. Then we show that this graph does not support exponential intersection tails.

2 Flows on $\mathcal{C}_\infty(\mathbf{Z}^2, p)$

We proceed using the method of Grimmett, Kesten, and Zhang. We will find a tree in $\mathcal{C}_\infty(\mathbf{Z}^2, p)$. The branches of this tree split into two at fairly regular intervals. With this tree we will associate a flow. Since we have control on how often the branches of the tree bifurcate we will be able bound the q energy of the flow. This method allows us to show that there exist flows on $\mathcal{C}_\infty(\mathbf{Z}^2, p)$ of finite q energy for all $q > 2$.

First we introduce some notation. A **path** P is a sequence of open directed edges where the end of one edge is the beginning of the next edge. A path P **connects** x and

y if x and y are endpoints of edges in P. We write $x \sim y$ if there exists an open path from x to y. We write $x \sim \infty$ if x is part of the unique infinite cluster. If $x \sim y$ then we let $D(x, y)$ be the length of the shortest open path from x to y. We use the taxicab metric on $\mathbf{Z}^2$, $|(x, y), (x', y')| = |x - x'| + |y - y'|$.

Let $\beta > 2$, $\rho \geq 1$, and b be constants which will be defined later. Let $a = 20\rho b$. Let $X_k(i) = (\beta^k, i(\beta/2)^k)$ for all i, $-2^{k-1} < i \leq 2^{k-1}$, and $k \geq 1$. Define

$$\bar{X}_k(i) = .5(X_{k+1}(2i) + X_{k+1}(2i-1)).$$

The tree in the percolation cluster that we will construct will have the following properties. For each i and k there will be one branch going from near $X_k(i)$ to near $\bar{X}_k(i)$. This branch will then bifurcate near $\bar{X}_{k+1}(2i)$. One of the branches goes toward $X_{k+1}(2i)$, while the other branch goes toward $X_{k+1}(2i-1)$.

Define $L(u, v)$ to be the elements in $\mathbf{Z}^2$ which are within $\sqrt{2}$ of the line segment joining u and v. Let $B(k)$ be all $(x, y) \in \mathbf{Z}^2$ such that $|x| + |y| \leq k$. Define

$$T_k(i) = B(ak) + \left(L(X_k(i), \bar{X}_k(i)) \cup L(X_{k+1}(2i), X_{k+1}(2i-1))\right),$$

where $+$ represents Minkowski addition.

Lemma 2.1 *There exists a function $K(a)$ such that*

$$T_k(i) \cap T_k(j) = \emptyset$$

for all $i \neq j$ and $k > K(a)$. We also have that

$$T_k(i) \cap T_{k+1}(j) = \emptyset$$

for all $j \neq 2i$ or $j \neq 2i - 1$ and $k > K(a)$.

Proof: If $T_k(i) \cap T_k(j)$ is not empty the there exists a point in

$$L(X_k(i), \bar{X}_k(i)) \cup L(X_{k+1}(2i), X_{k+1}(2i-1))$$

which is within $2ak$ of a point in

$$L(X_k(j), \bar{X}_k(j)) \cup L(X_{k+1}(2j), X_{k+1}(2j-1)).$$

Without loss of generality this implies either $|X_k(i), X_k(j)| \le 2ak$ or $|X_{k+1}(2i), X_{k+1}(2j-1)| \le 2ak$. Those two distances are at least $(\beta/2)^k$. So the first condition is satisfied if $(\beta/2)^{K(a)} > 2aK(a)$. The second condition is also satisfied with the same choice of $K(a)$ for the same reason. ♯

Our next goal is to give a sufficient condition for there to be an open path from a point near $X_k(i)$ to a point near $X_{k+1}(2i)$. Our condition will also imply that the path lies entirely inside $T_k(i)$ and has length bounded by $C|X_k(i), X_{k+1}(2i)|$.

Find $y_1 = y_1(k,i)$ through $y_t = y_t(k,i)$ such that

1. $y_1 = X_k(i)$,
2. $|y_t - \bar{X}_k(i)| \le 1$
3. $4bk \le |y_u, y_{u+1}| \le 8bk$, for all $1 \le u \le t-1$
4. $y_u \in L(X_k(i), \bar{X}_k(i))$, for all $1 \le u \le t-1$ and
5. $t \le \beta^{k+1}/4bk$.

Define $E_u = E_u(k,i)$ to be the event that

1. there exists $z_1 \in y_u + B(bk)$ such that $z_1 \sim \infty$, and
2. any two points $z_2 \in (y_u + B(bk))$ and $z_3 \in (y_{u+1} + B(bk))$ which are connected have $D(z_2, z_3) \le ak/2$. This implies that the shortest path connecting z_2 and z_3 lies in $y_u + B(ak) \subset T_k(i)$.

If all the E_u hold then there is a path from near $X_k(i)$ to near $\bar{X}_k(i)$ inside $T_k(i)$.

In a similar manner we can find y'_1 through $y'_{t'}$ such that

1. $y'_1 = X_{k+1}(2i-1)$,
2. $y'_t = X_{k+1}(2i)$,
3. $4bk \le |y'_u, y'_{u+1}| \le 8bk$, for all $1 \le u \le t'-1$
4. $y'_u \in L(X_{k+1}(2i-1), X_{k+1}(2i))$, for all $1 \le u \le t'-1$,

5. $t' \leq 5\rho\beta^{k+1}/bk$, and

6. there exist u and u' such that $y_u = y'_{u'}$.

Define $E'_u = E'_u(k,i)$ to be the event that

1. there exists $z_1 \in y'_u + B(bk)$ such that $z_1 \sim \infty$, and

2. any two points in $z_2 \in (y'_u + B(bk))$ and $z_3 \in (y'_{u+1} + B(bk))$ which are connected have $D(z_2, z_3) \leq ak/2$. This implies that the shortest path connecting z_2 and z_3 lies in $y'_u + B(ak) \subset T_k(i)$.

Define $E_k(i) = (\cap_u E_u(k,i)) \cap (\cap_u E'_u(k,i))$. In this next lemma we show that if $E_k(i)$ holds for one k and i then there exist a path from near $X_k(i)$ to near $X_{k+1}(2i)$ and a path from near $X_k(i)$ to near $X_{k+1}(2i-1)$. We also show that if $E_k(i)$ holds for all k sufficiently large and all i then there exists a tree with the desired properties in $\mathcal{C}_\infty(\mathbf{Z}^2, p)$. This implies that there exists a flow of finite q energy for all $q > 1 + \log_2 \beta$.

Lemma 2.2 *If there exists a K such that $E_k(i)$ holds for all $k \geq K$ and i then there exists a flow on $\mathcal{C}_\infty(\mathbf{Z}^2, p)$ with finite q energy for all $q > 1 + \log_2 \beta$.*

Proof: It causes no loss of generality to assume that $K > K(a)$. We construct a multisource flow that has a source near each $X_K(i)$. We will show that the multisource flow has finite energy. Thus there exists a flow with finite energy. Since $E_k(i)$ holds for each $k \geq K$ and i it is possible to pick points $p_k(i)$ such that $p_k(i) \in X_k(i) + B(bk)$ and $p_k(i) \sim \infty$. For each k and i it is possible to pick points $z_u \in y_u + B(bk)$ such that $z_u \sim \infty$. It is also possible to select $z'_u \in y'_u + B(bk)$ such that $z'_u \sim \infty$. We can also require that there exist u and u' such that $z_u = z'_{u'}, z_1 = p_k(i), z'_1 = p_{k+1}(2i-1)$, and $z'_t = p_{k+1}(2i)$. The second condition in the definition of E_u implies that $z_u \sim z_{u+1}$. Since all the E_u and E'_u hold we can piece together these paths to form paths $P_k(i)$ and $P'_k(i)$ such that

1. $P_k(i)$ which connects $p_k(i)$ and $p_{k+1}(2i)$

2. $P'_k(i)$ which connects $p_k(i)$ and $p_{k+1}(2i-1)$

3. $P_k(i), P'_k(i) \subset T_k(i)$, and

4. $|P_k(i)|, |P'_k(i)| \leq (ak/2)(\beta^{k+1}/4bk) = 5\rho\beta^{k+1}/2$.

The union of the $P_k(i)$ and the $P'_k(i)$ does not necessarily form a tree. However it is easy to remove branches so that it does form a tree. Instead of doing this we use the $P_k(i)$ and the $P'_k(i)$ to define a flow directly. For each edge e assign it mass

$$f(e) = \sum_{k,i} \frac{1}{2^k} \left(I_{P_k(i)}(e) + I_{P'_k(i)}(e) \right).$$

Lemma 2.1, the fact that $K \geq K(a)$, and condition 3 on the $P_k(i)$ implies that if $e \in P_k(i) \cup P'_k(i)$ and $k > K(a)$ then $f(e) \leq 4(2^{-k})$. If

$$\sum_{k>K} \sum_i \left(\sum_{e \in P_k(i)} f(e)^q + \sum_{e \in P'_k(i)} f(e)^q \right) < \infty$$

then $\sum f(e)^q < \infty$ is finite. Thus the following calculation shows f has finite q energy.

$$\begin{aligned}
\sum_{k>K} \sum_i \left(\sum_{e \in P_k(i)} f(e)^q + \sum_{e \in P'_k(i)} f(e)^q \right) &\leq \sum_{k>K} \sum_i (4(2^{-k}))^q 5\rho\beta^{k+1} \\
&\leq \sum_{k>K} \sum_i C2^{-kq}\beta^k \\
&\leq \sum_{k>K} C2^{-k(q-\log_2 \beta)} 2^{k+1} \\
&\leq \sum_{k>K} 2C2^{-k(q-\log_2 \beta - 1)} \\
&< \infty.
\end{aligned}$$

♯

Now we show that with probability one there exists a K so that $E_k(i)$ holds for all $k > K$ and all i. To do this we need to bound the probabilities of E_u and the $E_{u'}$. This requires one two theorems. The first follows from the work of Russo [13], Seymour and Welsh [14], and Kesten [9].

Theorem 2.1 *Given $p > .5$ there exists C_1 and α_1 so that* $\mathbf{P}(B(k) \not\sim \infty) < C_1 2^{-\alpha_1 k}$.

The second is due to Antal and Pisztora [2].

Theorem 2.2 *[2] Let $p > .5$. Then there exists $\rho = \rho(p) \in [1, \infty)$, and constants C_2, and $\delta > 0$ such that for all $|y|$*

$$\mathbf{P}(0 \sim y, D(0, y) > \rho|y|) < C_2 2^{-\delta|y|}.$$

Lemma 2.3 *There exists an $\alpha > 0$ such that for all k*

$$\mathbf{P}(E_u(k,i)) > 1 - C_3 2^{-\alpha bk}.$$

Proof: By Theorem 2.1 the probability that condition 1 in the definition of $E_u(k,i)$ does not hold is less than $C_1 2^{-\alpha_1 bk}$. If condition 2 is not true then there exist two points that are connected but the distance between them is large. Theorem 2.2 implies that

$$\mathbf{P}\left(\exists\ z_2 \text{ and } z_3 \text{ such that } D(z_2,z_3) > ak/2 = 10\rho bk \geq \rho|z_2 - z_3|\right) < (bk)^2 C_2 2^{-2\delta bk}.$$

Thus

$$\begin{aligned}\mathbf{P}(E_u(k,i)) &> 1 - C_1 2^{-\alpha_1 bk} - (bk)^2 C_2 2^{-2\delta bk} \\ &> 1 - C_3 2^{-\alpha bk}\end{aligned}$$

for an appropriate choice of C_3 and α. We also get the same bound for $\mathbf{P}(E'_u)$. ♯

Lemma 2.4 *For $p > .5$ there exists b, C_4, and $\alpha' > 1$ such that for all k and i*

$$\mathbf{P}(E_k(i)) \geq 1 - C_4 2^{-\alpha' k}.$$

Proof: Choose b large enough so that $\alpha' = \alpha b - \log_2 \beta > 1$.

$$\begin{aligned}\mathbf{P}(E_k(i)) &\geq 1 - \sum_u \mathbf{P}(E_u(k,i)^C) - \sum_u \mathbf{P}(E'_u(k,i)^C) \\ &\geq 1 - 5\rho\beta^{k+1} C_3 2^{-\alpha bk} \\ &\geq 1 - C_4 2^{k \log_2 \beta - \alpha bk} \\ &\geq 1 - C_4 2^{-\alpha' k}.\end{aligned}$$

♯

Proof of Theorem 1.1: Given $p > .5$ and $q > 2$ choose β so that $\beta > 2$ and $q > 1 + \log_2 \beta$. The previous lemma implies that $\lim_n \mathbf{P}(\cap_{i,k>n} E_k(i)) = 1$. Thus the Borel-Cantelli lemma and Lemma 2.2 there exists a flow of finite q energy with probability 1. ♯

3 A Refinement

In this section we will be concerned with a generalization of the energy of a flow. For any function H we define the H energy of a flow f as

$$\mathcal{E}_H(f) = \sum_e H(f(e)).$$

We will be working with a special class of functions. For any $\alpha > 0$, let

$$H_{2,\alpha}(u) = u^2/[\log(1+u^{-1})]^\alpha$$

for $u > 0$ and $H_{2,\alpha}(0) = 0$. The main result of this section is that for $d = 2$ there exists a flow of finite $H_{2,\alpha}$ energy on $\mathcal{C}_\infty(\mathbf{Z}^2, p)$ for all $\alpha > 2$.

The proof is very similar to the previous section. Let $\beta > 1$, $\rho \geq 1$, and b be constants which will be defined later. Let $a = 20\rho b$. Define $t_k = 2^k k^\beta$. Let $X_k(i) = (t_k, (it_k)/2^k)$ for $|i| \leq 2^k$ and $k \geq 0$. Then define $T_k(i)$ as in section 2.

Lemma 3.1 *For each $\beta > 1$ there exists a function $K(a)$ such that*

$$T_k(i) \cap T_k(j) = \emptyset \tag{1}$$

for all $i \neq j$ and $k > K(a)$. We also have that

$$T_k(i) \cap T_{k+1}(j) = \emptyset \tag{2}$$

for all $j \neq 2i$ or $j \neq 2i-1$ and $k > K(a)$.

Proof: If $T_k(i) \cap T_k(j)$ is not empty the there exists a point in

$$L(X_k(i), \bar{X}_k(i)) \cup L(X_{k+1}(2i), X_{k+1}(2i-1))$$

which is within $2ak$ of a point in

$$L(X_k(j), \bar{X}_k(j)) \cup L(X_{k+1}(2j), X_{k+1}(2j-1)).$$

Without loss of generality this implies either $|X_k(i), X_k(j)| \leq 2ak$ or $|X_{k+1}(2i), X_{k+1}(2j-1)| \leq 2ak$. Those two distances are at least k^β. So the first condition is satisfied if $K(a)^\beta > 2aK(a)$. This can be achieved whenever $\beta > 1$. The second condition is also satisfied with the same choice of $K(a)$ for the same reason. ♯

Pick the $y_1, ..., y_t, y'_1, ..., y'_t$ as in section 2. This can be done with

$$t \leq t_{k+1}/4bk < 2^{k-1}(k+1)^{\beta}/bk.$$

Define E_u, E'_u, and $E_k(i)$ the same as in the previous section.

Lemma 3.2 *If $E_k(i)$ holds for all $k \geq K$ and all i then there exists a flow on $\mathcal{C}_\infty(\mathbf{Z}^2, p)$ with finite $H_{2,\alpha}$ energy for all $\alpha > 1+\beta$.*

Proof: Again we create a multisource flow with finite energy. We assume that $K > K(a)$. Since $E_k(i)$ holds for each k and i it is possible to pick points $p_k(i)$ such that $p_k(i) \in X_k(i) + B(bk)$ and $p_k(i) \sim \infty$. We can also define z_u and z'_u as in the previous section. Since $E_k(i)$ hold then there exists paths $P_k(i)$ and $P'_k(i)$ such that

1. $P_k(i)$ which connects $p_k(i)$ and $p_{k+1}(2i)$
2. $P'_k(i)$ which connects $p_k(i)$ and $p_{k+1}(2i-1)$
3. $P_k(i), P'_k(i) \subset T_k(i)$, and
4. $|P_k(i)|, |P'_k(i)| \leq (ak/2)(2^{k-1}(k+1)^{\beta}/bk) < 5\rho 2^k(k+1)^{\beta}$.

For each edge e assign it mass

$$f(e) = \sum_{k,i} \frac{1}{2^k}\Big(I_{P_k(i)}(e) + I_{P'_k(i)}(e)\Big).$$

Lemma 3.1 implies that if $e \in P_k(i) \cup P'_k(i)$ and $k > K(a)$ then

$$f(e) \leq 4(2^{-k}). \tag{3}$$

If

$$\sum_{i,k>K}\left(\sum_{e\in P_k(i)} H_{2,\alpha}(f(e)) + \sum_{e\in P'_k(i)} H_{2,\alpha}(f(e))\right) < \infty$$

then $\sum H_{2,\alpha}(f(e)) < \infty$. Thus the following calculation shows f has finite $H_{2,\alpha}$ energy.

$$\begin{aligned}
\sum_{k>K}\sum_{i}\left(\sum_{e\in P_k(i)} H_{2,\alpha}(f(e)) + \sum_{e\in P'_k(i)} H_{2,\alpha}(f(e))\right) &\leq \sum_{k>K}\sum_{i} \frac{(4(2^{-k}))^2}{[\log(1+(4(2^{-k}))^{-1})]^{\alpha}} 10\rho 2^k(k+1)^{\beta} \\
&\leq \sum_{k>K}\sum_{i} \frac{C(2^{-k})k^{\beta}}{[-\log(4(2^{-k}))]^{\alpha}}
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{k>K} C\frac{(2^{-k})k^{\beta}}{[k-2]^{\alpha}}2^{k+1}\\
&\le \sum_{k>K} C' k^{\beta-\alpha}\\
&< \infty
\end{aligned}
$$

♯

Proof of Theorem 1.2: Given $p > .5$ and $\alpha > 2$ choose β so that $\beta > 1$ and $\alpha > 1+\beta$. Choose b so that Lemma 2.4 implies that $\lim_n \mathbf{P}(\cap_{i,k>n} E_k(i)) = 1$. Thus by Lemma 3.2 and the Borel-Cantelli lemma there exists a flow of finite $H_{2,\alpha}$ energy with probability 1. ♯

4 A graph with transient percolation clusters that do not admit exponential intersection tails

In this section we will construct a graph G. Using the results of the previous section we are able to show that G has transient percolation clusters. Then we will show that G does not admit exponential intersection tails.

Let $b_i = 2^{2^{\lfloor i/5 \rfloor}}$. Before we define G we define

$$T = \{(x,y) \mid x \in \mathbf{N}, y \in \{0,1\}^{\mathbf{N}}, y_i = 0 \text{ for all } i \text{ such that } b_i \ge x\}$$

to be the tree whose branches split at distance b_i from the root. Thus a vertex (x,y) is distance x from the root. The vertex (x,y) is connected to the vertex $(x+1,y)$. If $x = b_i$ for some i then the vertex (x,y) is also connected to the vertex $(x+1,\tilde{y})$, where $\tilde{y}_i = 1$ and $\tilde{y}_j = y_j$ for all $j \ne i$. At distance d from the root of T there are $\mathcal{O}(\log d)^5$ branches. Let

$$G = \{(x,y,z) \mid (x,y) \in T, z \in \mathbf{Z}, \text{ and } |z| \le .6x\}.$$

Theorem 4.1 *For all $p > 1/2$ simple random walk on the infinite percolation clusters on G is a.s. transient.*

Proof: We use the construction in the previous section with $\beta = 1.5$. To each $y \in \{0,1\}^{\mathbf{N}}$ corresponds

$$G_y = \{(x,y',z) \in G \mid y_i = y'_i \text{ for all } i \text{ such that } x < b_i \text{ and } y'_i = 0 \text{ else}\},$$

a subset of G which looks like a wedge of $\mathbf{Z}^2$. By the previous section we have with positive probability exhibited a flow, f_y, on the infinite percolation cluster of G_y. This flow has finite $H_{2,3}$ energy. Define the multisource flow $F(e) = \int f_y(e)dm$, where m is $(1/2, 1/2)$ product measure on $\{0,1\}^{\mathbf{N}}$.

Denote by $\bar{T}_k(i)$ all branches (x, y, z) such that $(x, z) \in T_k(i)$. Now we calculate the energy of F.

$$\begin{aligned}
\sum_{k>K} \sum_i \sum_{e \in \bar{T}_k(i)} F(e)^2 &= \sum_{k>K} \sum_i \sum_{e \in \bar{T}_k(i)} \left(\int f_y(e)dm \right)^2 \\
&= \sum_{k>K} \sum_i \sum_{e \in \bar{T}_k(i)} \left(\int_{B_e} f_y(e)dm \right)^2
\end{aligned}$$

where $B_e = (y' \mid y'_i = y_i \text{ for all } b_i < x)$ for $e = (x, y, z)$. The last equality is true because the definition of $f_{y'}$ implies that $f_{y'}(e) = 0$ for all $y' \notin B_e$. Jensen's inequality generates the inequality

$$\left(\int_X f dm \right)^2 \leq (m(X)) \left(\int_X (f)^2 dm \right).$$

Since $m(B_e) < C(\log |e|)^{-5}$

$$\begin{aligned}
\sum_{k>K} \sum_i \sum_{e \in \bar{T}_k(i)} F(e)^2 &\leq \sum_{k>K} \sum_i \sum_{e \in \bar{T}_k(i)} \frac{C}{k^5} \int_{B_e} f_y(e)^2 dm \\
&\leq \sum_{k>K} \sum_i \sum_{e \in \bar{T}_k(i)} \frac{C}{k^5} \int f_y(e)^2 dm \\
&\leq \sum_{k>K} \frac{C}{k^5} \sum_i \int \left(\sum_{e \in \bar{T}_k(i)} f_y(e)^2 dm \right) \\
&\leq \sum_{k>K} \frac{C}{k^5} \sum_i (4(2^{-k}))^2 10 \rho 2^k (k+1)^3 \qquad (4) \\
&\leq \sum_{k>K} \frac{C'}{k^2} \\
&< \infty.
\end{aligned}$$

Line (4) is true by equation (3) and condition 4 in the proof of Lemma 3.2. Thus if $p > .5$ infinite percolation clusters on G support flows of finite energy and are transient. $\sharp$

Now we show that there is no measure on paths on G with exponential intersection tails. We do this as follows. First we define a function $f(\psi, \varphi)$. We show that if μ is a measure on paths with exponential intersection tails then $\int f d(\mu \times \mu)$ is finite. Then we show that for any measure on paths on G that the integral must be infinite. Thus the graph G does not admit exponential intersection tails. It causes no loss of generality to assume that μ is supported on transient paths.

Let $|\psi \cap \varphi|$ be the number of edges that ψ and φ have in common. Let $\chi_e(\psi, \varphi)$ be the event that $e \in \psi \cap \varphi$. Define

$$f(\psi, \varphi) = \sum_{i=1}^{|\psi \cap \varphi|} i^5.$$

If $e \in \psi$ let $l(e)$ be the largest number such that $\psi_{l(e)} = e$. Let $\psi|_i$ be the path given by edges $\psi_{i+1}, \psi_{i+2}, \ldots$ Notice that

$$f \geq \sum_e \chi_e(\psi, \varphi)(|\psi \cap \varphi| - |\psi|_{l(e)} \cap \varphi|)^5. \quad (5)$$

Lemma 4.1 *If μ is a measure with exponential intersection tails then*

$$\int f d(\mu \times \mu) < \infty.$$

Proof: Because μ has exponential intersection tails there exists constants C and $\theta < 1$ such that $\mu \times \mu(|\psi \cap \varphi| \geq i) \leq C\theta^i$. Thus

$$\begin{aligned}
\int f d(\mu \times \mu) &= \int \sum_{i=1}^{|\psi \cap \varphi|} i^5 d(\mu \times \mu) \\
&= \sum_{i=1}^{\infty} i^5 \int \chi_{|\psi \cap \varphi| \geq i} d(\mu \times \mu) \\
&\leq \sum_i i^5 \mu \times \mu(|\psi \cap \varphi| \geq i) \\
&\leq \sum_i i^5 C\theta^i \\
&< \infty.
\end{aligned}$$

♯

This next lemma is the main tool we will use to show that $\int f d(\mu \times \mu) = \infty$ for any measure μ on paths in G.

Lemma 4.2 *There exists C such that for any μ and $e \in G$*

$$\mathbf{E}(|\psi \cap \varphi| - |\psi_{l(e)} \cap \varphi| \mid e \in \psi \cap \varphi) > C \log |e|.$$

Proof: Let $E_n(e)$ be the set of all edges at distance n from e. Since $e = \varphi_{l(e)}$ and μ is supported on transient paths then there exists a $k > l(e)$ so that $\varphi_k \in E_n(e)$. We can rewrite this as

$$\sum_{e' \in E_n(e)} \mathbf{P}(e' \in \psi \setminus \psi|_{l(e)} \mid e \in \psi) \geq 1.$$

The expected number of intersections of $\psi \setminus \psi|_{l(e)}$ and φ in $E_n(e)$ is

$$\begin{aligned} \mathbf{E}(|\psi \cap \varphi \cap E_n(e)| - |\psi|_{l(e)} \cap \varphi \cap E_n(e)| \mid e \in \psi \cap \varphi) &\geq \sum_{e' \in E_n(e)} \mathbf{P}(e' \in \psi \setminus \psi|_{l(e)} \mid e \in \psi)^2 \\ &\geq 1/|E_n(e)|. \end{aligned}$$

The last inequality is true because of the Cauchy-Schwartz inequality.

Let $e = (x, y, z), e' = (x', y', z')$ and $e' \in E_n(e)$. Define $|e| = x + |z|$. Then $.5|e| < x \leq |e|$. If $n \leq |e|/10$ then $.3x < x' < 1.2x$. As any interval $(j, 4j)$ has at most one b_i there can be at most 3 possible values for z'. For each x' and z' there are at most two possible values for y'. Thus for any $e \in G$ and $n \leq |e|/10$ we have that $|E_n(e)| \leq 12n$.

Putting these two facts together we get that

$$\mathbf{E}(|\psi \cap \varphi| - |\psi|_{l(e)} \cap \varphi| \mid e \in \psi \cap \varphi) > \sum_{n=1}^{|e|/10} 1/12n > C \log |e|.$$

♯

Theorem 4.2 *G does not admit exponential intersection tails.*

Proof: We argue by contradiction. If there were a measure μ with exponential intersection tails then Lemma 4.1 says that $\int f d(\mu \times \mu) < \infty$. The following calculation shows that this integral must be infinite.

$$\begin{aligned} \int f d(\mu \times \mu) &\geq \int \sum_e \chi_e(\psi, \varphi)(|\psi \cap \varphi| - |\psi|_{l(e)} \cap \varphi|)^5 d(\mu \times \mu) && (6) \\ &\geq \sum_e \int \chi_e(\psi, \varphi) \mathbf{E}(|\psi \cap \varphi| - |\psi|_{l(e)} \cap \varphi| \mid e \in \psi \cap \varphi)^5 d(\mu \times \mu) && (7) \\ &\geq \sum_e \sum_{|e|=j} \int \chi_e(\psi, \varphi)(C \log j)^5 d(\mu \times \mu) && (8) \end{aligned}$$

$$
\begin{aligned}
&\geq \sum_j (C\log j)^5 \sum_{|e|=j} (\mu\times\mu)(e\in\psi, e\in\varphi) \\
&\geq \sum_j (C\log j)^5 \sum_{|e|=j} \mu(e\in\psi)^2 \\
&\geq \sum_j (C\log j)^5 \frac{1}{C'j(\log j)^5} \qquad (9) \\
&\geq \sum_j \frac{C''}{j} \\
&\geq \infty
\end{aligned}
$$

Line (6) comes from line (5). Line (7) follows from Jensen's inequality. Line (8) follows from Lemma 4.2. Line (9) is true because the number of e such that $|e| = i$ is bounded by $C'i(\log i)^5$. Thus μ does not have exponential intersection tails. ♯

Proof of Theorem 1.3: Theorem 1.3 is a combination of Theorems 4.1 and 4.2. ♯

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