Coexistence for Richardson type competing spatial growth models

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Abstract

We study a large family of competing spatial growth models. In these the vertices in $\mathbb{Z}^d$ can take on three possible states {0,1,2}. Vertices in states 1 and 2 remain in their states forever, while vertices in state 0 which are adjacent to a vertex in state 1 (or state 2) can switch to state 1 (or state 2). We think of the vertices in states 1 and 2 as infected with one of two infections while the vertices in state 0 are considered uninfected. In this way these models are variants of the Richardson model. We start the models with a single vertex in state 1 and a single vertex is in state 2. We show that with positive probability state 1 reaches an infinite number of vertices and state 2 also reaches an infinite number of vertices. This extends results and proves a conjecture of Häggström and Pemantle [5]. The key tool is applying the ergodic theorem to stationary first passage percolation.

1 First Passage Percolation

In this paper we study a class of competing spatial growth models by first studying stationary first passage percolation and then applying our results to the spatial growth models. In first passage percolation every edge in a graph is assigned a non-negative number. This is interpreted as the time it takes to move across the edge. This model was introduced by Hammersley and Welsh [6]. See [7] for an overview of first passage percolation.

Let μ be a stationary measure on $[0,\infty)^{\text{Edges}(\mathbb{Z}^d)}$ and let ω be a realization of μ. For any x and y we define the **passage time from** x **to** y, $\tau(x,y)$, by

$$\tau(x,y) = \inf \sum \omega(v_i, v_{i+1})$$

where the sum is taken over all of the edges in the path and the inf is taken over all paths connecting x to y.

The most basic result from first passage percolation is the shape theorem. We let $\mathbf{0} = (0,\ldots,0)$ and $\mathbf{1} = (1,0,\ldots,0)$. Define

$$S(t) = \{x : \ \tau(\mathbf{0}, x) \le t\}$$

and

$$\bar{S}(t) = S(t) + \left[-\frac{1}{2}, \frac{1}{2}\right]^d.$$

The shape theorem says that there is a nonempty set S such that $\frac{\bar{S}(t)}{t}$ converges to S a.s.

Theorem 1. *[1] Let μ be stationary and ergodic, with the distribution on any edge have finite $d+\epsilon$ moment for $\epsilon > 0$. There exists a set S which is nonempty, convex, and symmetric about reflection through the origin such that for every $\epsilon > 0$ there exists a T such that for all $t > T$*

$$P\left((1-\epsilon)S < \frac{\bar{S}(t)}{t} < (1+\epsilon)S\right) > 1 - \epsilon.$$

This theorem is a consequence of Kingman's subadditive ergodic theorem. It is the only property of first passage percolation that we need. In general little is known about the shape of S other than it is convex and symmetric. Cox and Durrett have shown that there are nontrivial product measures such that the boundary of S contains a flat piece [2]. However for any compact nonempty convex set S there exist a stationary measure μ such that the shape for μ is S [4].

Another widely studied aspect of first passage percolation are geodesics. A **geodesic** is a path $G = \{v_0, v_1, \dots\}$ such that

$$\tau(v_m, v_n) = \sum_{i=m}^{n-1} \omega(v_i, v_{i+1})$$

for any $m < n$. We let $G^\omega(x, y) = G(x, y)$ be the union of all geodesics connecting x and y. Define

$$\Gamma(x) = \cup_{y \in \mathbb{Z}^d} \{e : e \in \text{Edges}(\mathbb{Z}^d) \text{ and } e \in G(x, y)\}.$$

We refer to this as the **tree of infection of** x. We define $K(\Gamma(x))$ to be the number of topological ends in $\Gamma(x)$. This is also the number of infinite self avoiding paths in $\Gamma(x)$ that start at x.

Newman has conjectured that for a large class of μ, $|K(\Gamma(\mathbf{0}))| = \infty$ a.s. [8] Häggström and Pemantle proved that if $d = 2$ and μ is the i.i.d with exponential distribution then with positive probability $|K(\Gamma(\mathbf{0}))| > 1$. Newman has proved that if μ is i.i.d. and S has certain properties then $|K(\Gamma(\mathbf{0}))| = \infty$ a.s. [8] Although these conditions are plausible there are no known measures μ with S that satisfy these conditions.

Now we will introduce some more notation which will let us list the conditions that we place on μ. We say that the configuration ω has **unique geodesics** if for all $x, y \in \mathbb{Z}^d$ there exists a unique geodesic from x to y. If there exists a unique geodesic between x and y we denote it by $G(x, y)$. The configuration ω has **unique passage times** for all x and $y \neq z$

$$\tau(x, y) \neq \tau(x, z).$$

For any ω we let $\mu_\omega^{(\mathbf{0},\mathbf{1})}$ be the conditional distribution of μ on the edge $(\mathbf{0},\mathbf{1})$ given that $\omega'(v,w) = \omega(v,w)$ for all edges except $(\mathbf{0},\mathbf{1})$. We say that μ has **finite energy** if for any set $A \subset \mathbb{R}$ such that $\mu\{\omega(\mathbf{0},\mathbf{1}) \in A\} > 0$ and almost every ω, $\mu_\omega^{(\mathbf{0},\mathbf{1})}\{\omega' : \omega'(0,1) \in A\} > 0$.

As μ is a stationary measure we can study its ergodic theoretical properties. For any $v \in \mathbb{Z}^d$ define the shift map $T^v : [0,\infty)^{\text{Edges}(\mathbb{Z}^d)} \to [0,\infty)^{\text{Edges}(\mathbb{Z}^d)}$ by

$$T^v(\omega)(j) = \omega(j+v)$$

for all $j \in \text{Edges}(\mathbb{Z}^d)$. The measure μ is **totally ergodic** if for all $v \in \mathbb{Z}^d$ the action (μ, T^v) is ergodic.

Now we are ready to define the class of measures that we will work with. We say that μ is **good** if

1. μ is totally ergodic,
2. μ has all the symmetries of $\mathbb{Z}^d$,
3. the distribution of μ on any edge has finite $d+\epsilon$ moment for some $\epsilon > 0$
4. μ has finite energy
5. $\mu_\omega^{(\mathbf{0},\mathbf{1})}$ is an absolutely continuous measure with support $[0,\infty)$ a.s., and
6. μ produces a shape S which is bounded.

Note that conditions 2, 4 and 5 imply that μ has unique geodesics and unique passage times. These conditions were chosen to make the arguments as easy as possible and could be made more general. All that is essential for the argument to show that there are at least two disjoint infinite geodesics is that μ is totally ergodic and that Lemma 1 and Corollary 1 below are satisfied. The conditions 2, 4 and 5 are used to show that coexistence occurs with positive probability. Throughout the rest of the paper we will assume that μ is good. Unfortunately there is no general necessary and sufficient condition to determine when the shape S is bounded. See [4] for examples.

2 Spatial Growth Models

Now we explain the relationship between first passage percolation and our competing growth models. For any $\omega \in [0,\infty)^{\text{Edges}(\mathbb{Z}^d)}$ with unique passage times and any $x \neq y \in \mathbb{Z}^d$ we can project it to $\tilde{\omega}_{x,y} \in \left(\{0,1,2\}^{\mathbb{Z}^d}\right)^{[0,\infty)}$ by

$$\tilde{\omega}_{x,y}(z,t) = \begin{cases} 2 & \text{if } \tau(x,z) \le t \text{ and } \tau(x,z) < \tau(y,z); \\ 1 & \text{if } \tau(y,z) \le t \text{ and } \tau(x,z) > \tau(y,z); \\ 0 & \text{else.} \end{cases}$$

If μ has unique passage times a.s. then μ projects onto a measure on $\left(\{0,1,2\}^{\mathbb{Z}^d}\right)^{[0,\infty)}$. It is clear that the models start with a single vertex in state 1 and a single vertex is in state 2. Vertices in states 1 and 2 remain in their states forever, while vertices in state 0 which are adjacent to a vertex in state 1 (or state 2) can switch to state 1 (or state 2). We think of the vertices in states 1 and 2 as infected with one of two infections while the vertices in state 0 are considered uninfected. In this way these models are variants of the Richardson model.

As each $z \in \mathbb{Z}^d$ eventually changes to state 1 or 2 and then stays in that state for the rest of time, we can speak of the limiting configuration. There are two possible outcomes. The first is coexistence or mutual unbounded growth. If this occurs then the limiting configuration has infinitely many z in state 1 and infinitely many z in state 2. The other outcome is domination. If this happens then in the limiting configuration there are only finitely many vertices in that state and all but finitely many vertices are in the other state.

For many measures μ (for example if μ is i.i.d. with nontrivial marginals) then it is easy to prove that domination occurs with positive probability. But it is much more difficult to show that coexistence occurs with positive probability. More precisely we define $\mathrm{C}(x,y)$ to be the event that

$$|\{z : \lim_{t\to\infty} \tilde{\omega}_{x,y}(z) = 1\}| = |\{z : \lim_{t\to\infty} \tilde{\omega}_{x,y}(z) = 2\}| = \infty.$$

We refer to this event as **coexistence** or **mutual unbounded growth**. Our main result is that with positive probability coexistence occurs.

Theorem 2. *If μ is good then*

$$P(C(\mathbf{0},\mathbf{1})) > 0.$$

This proves a conjecture of Häggström and Pemantle [5]. They proved this theorem in the case that $d = 2$ and μ is i.i.d. with exponential distribution. Garet and Marchand have given a different proof of Theorem 2 [3]. Their method follows more closely the approach taken by Häggström and Pemantle.

3 Outline

In this section we outline the proof of our main result. For any $x, y \in \mathbb{Z}^d$ and infinite geodesic $G = (v_0, v_1, v_2, \dots)$ we can define

$$B_G^\omega(x, y) = B_G(x, y) = \lim_{n\to\infty} \tau(x, v_n) - \tau(y, v_n).$$

To see the limit exists first note that

$$\begin{aligned} B_G(x, y) &= \lim_{n\to\infty} \tau(x, v_n) - \tau(y, v_n) \\ &= \lim_{n\to\infty} \tau(x, v_n) - \tau(v_0, v_n) + \tau(v_0, v_n) - \tau(y, v_n) \\ &= \lim_{n\to\infty} (\tau(x, v_n) - \tau(v_0, v_n)) + \lim_{n\to\infty} (\tau(v_0, v_n) - \tau(y, v_n)). \end{aligned}$$

As G is a geodesic the two sequences in the right hand side of the last line are bounded and monotonic so they converge. Thus $B_G(x, y)$ is well defined. If for a given ω and all $x, y \in \mathbb{Z}^d$ the function $B_G(x, y)$ is independent of the choice of infinite geodesic G then we can define the Busemann function

$$B(x, y) = B^\omega(x, y) = B_G^\omega(x, y).$$

The main step in our proof is Lemma 4, which states that the probability that $\{B(x, y)\}_{x,y\in\mathbb{Z}^d}$ is well defined is 0.

We will work by contradiction to prove Lemma 4. In Lemmas 2 and 3 we assume that $\{B(x, y)\}_{x,y\in\mathbb{Z}^d}$ is well defined a.s. and then apply the ergodic theorem to $\{B(x, y)\}_{x,y\in\mathbb{Z}^d}$. Then in Lemma 4 we show that the conclusions of Lemma 3 generate a contradiction with the shape theorem. Thus with positive probability there are vertices x and y and distinct geodesics $G_0 = G_0(\omega)$ and $G_1 = G_1(\omega)$ such that

$$B_{G_0}(x, y) \neq B_{G_1}(x, y).$$

From this point a short argument allows us to conclude that coexistence is possible with positive probability.

4 Proof

The heart of the proof is applying the ergodic theorem to the Busemann function. This is done in Lemmas 2 and 3. We start by showing that the symmetry of μ implies that the expected value of the Busemann function is 0.

Lemma 1. *If $\{B(x,y)\}_{x,y\in\mathbb{Z}^d}$ is well defined a.s. then for all $v \in \mathbb{Z}^d$*

$$\mathbf{E}(B(\mathbf{0},v)) = 0.$$

Proof. By symmetry of μ we have that $\mathbf{E}(B(\mathbf{0},\mathbf{1})) = \mathbf{E}(B(\mathbf{1},\mathbf{0}))$. Combining this with the fact that $B(\mathbf{0},\mathbf{1}) + B(\mathbf{1},\mathbf{0}) = 0$ proves the lemma. □

Now we apply the ergodic theorem to $B(\mathbf{0},v)$.

Lemma 2. *If $\{B(x,y)\}_{x,y\in\mathbb{Z}^d}$ is well defined a.s. then for all $v \in \mathbb{Z}^d$ and $\epsilon > 0$ there exists M such that*

$$P\left(|B(\mathbf{0},mv)| < \epsilon m \text{ for all } m > M\right) > 1-\epsilon.$$

Proof. First rewrite $B(\mathbf{0},mv)$ as follows.

$$\begin{aligned} B(\mathbf{0},mv) &= B(\mathbf{0},v) + B(v,2v) + \cdots + B((m-1)v,mv) \\ B(\mathbf{0},mv) &= B^{\omega}(\mathbf{0},v) + B^{T^{v}(\omega)}(\mathbf{0},v) + \cdots + B^{T^{(m-1)v}(\omega)}(\mathbf{0},v) \\ B(\mathbf{0},mv) &= \sum_{j=0}^{m-1} B^{T^{jv}(\omega)}(\mathbf{0},v) \qquad (1) \end{aligned}$$

As μ is good it is totally ergodic and the action (T^v,μ) is ergodic. Thus by line (1) and Lemma 1 the claim is a consequence of the ergodic theorem. □

We now strengthen this lemma by using the following corollary of shape theorem. For $x \in \mathbb{Z}^d$ we let $|x| = |x_1| + |x_2| + \cdots + |x_d|$.

Corollary 1. *There exist $0 < k_1 < k_2 < \infty$ such that for every $\epsilon > 0$ there exists an N such that*

$$P\left(k_1 < \frac{\tau(\mathbf{0},x)}{|x|} < k_2 \text{ for all } x \text{ such that } |x| > N\right) > 1-\epsilon.$$

Proof. The existence of k_2 is due to the fact that the set S (from Theorem 1) is nonempty. The existence of k_1 follows because one of the requirements of μ being good is that S is bounded. □

Lemma 3. *If $\{B(x,y)\}_{x,y\in\mathbb{Z}^d}$ is well defined a.s. then for any $\epsilon > 0$ there exists N such that if $n > N$ then*

$$P\left(\frac{B(\mathbf{0},x)}{|x|} < \epsilon \text{ for all } x \text{ such that } |x| = n\right) > 1-\epsilon.$$

Proof. Given $\epsilon > 0$ pick vectors $v_1, v_2, \ldots, v_j$ such that $|v_1| = |v_2| = \cdots = |v_j|$ and for all x sufficiently large there exists $i \in \{1,2,\ldots,j\}$ and $m \in \mathbb{N}$ such that

$$|x - mv_i| < \epsilon|x| \text{ and } m|v_i| \le |x|.$$

For all x and y we have that

$$B(\mathbf{0},x) = B(\mathbf{0},y) + B(y,x).$$

This implies that for any x and y

$$B(\mathbf{0},x) \leq B(\mathbf{0},y) + \tau(y,x).$$

For any n let m be the largest integer such that $m|v_i| \leq n$. (This is independent of i.) Thus if there exists x with $|x| = n$ and $\frac{B(\mathbf{0},x)}{|x|} \geq \epsilon$ then there exists i such that either

1. $B(\mathbf{0}, mv_i) \geq \epsilon n/2 = \epsilon|x|/2$, or
2. $|x - mv_i| < \epsilon|x|/2k_2$ and $\tau(x, mv_i) \geq \epsilon|x|/2$.

(The constant k_2 is from Corollary 1.)

By Lemma 2 there exists M such that

$$P(\text{there exists } m > M \text{ and } i \in \{1,2,\ldots,j\} \text{ such that } B(\mathbf{0},mv_i) > 2\epsilon m|v_i|/3 > \epsilon n/2) < 2\epsilon/3.$$

Thus the probability of the first event is less than $2\epsilon/3$ if n is sufficiently large.

By Corollary 1 there exists L such that for any $l > L$

$$P(\text{there exists } z \text{ with } |z| \leq l \text{ and } \tau(\mathbf{0},z) \geq k_2 l) < \epsilon/3j.$$

Applying this with each mv_i in place of $\mathbf{0}$ and $\epsilon n/2k_2$ in place of l we get that the probability of the second event is less than $\epsilon/3$ if n is sufficiently large. Thus for any $\epsilon > 0$ we get N so that if $n > N$ we get that

$$P\left(\text{there exists } x \text{ such that } |x| = n \text{ and } \frac{B(\mathbf{0},x)}{|x|} \geq \epsilon\right) < \epsilon$$

which proves the lemma. □

Next we show that this generates a contradiction with the shape theorem.

Lemma 4. *$P(\{B(x,y)\}_{x,y\in\mathbb{Z}^d}$ is well defined$) = 0$.*

Proof. We work by contradiction. Suppose that with positive probability $\{B(x,y)\}_{x,y\in\mathbb{Z}^d}$ is well defined. The Busemann function being well defined is a shift invariant event which, by the ergodicity of μ, implies that $\{B(x,y)\}_{x,y\in\mathbb{Z}^d}$ is well defined a.s. and the conclusions of Lemma 3 apply. Pick $\epsilon < \frac{1}{3}\min(k_1, 1)$, where k_1 comes from Corollary 1. By the choice of ϵ and Corollary 1 we have that there exists N such that for all $n > N$

$$P\left(\frac{\tau(\mathbf{0},x)}{|x|} > 2\epsilon \text{ for all } x \text{ such that } |x| = n\right) > \frac{2}{3}. \tag{2}$$

By Lemma 3 there exists $n > N$ such that

$$P\left(\frac{B(\mathbf{0},x)}{|x|} < \epsilon \text{ for all } x \text{ such that } |x| = n\right) > \frac{2}{3}. \tag{3}$$

However there exists at least one infinite geodesic $G = (\mathbf{0}, v_1, v_2, \dots)$ which begins at $\mathbf{0}$. (The choice of G is immaterial.) For all n there exists k such that $|v_k| = n$. For any k we have that $B(\mathbf{0}, v_k) = \tau(\mathbf{0}, v_k)$. This shows that lines (2) and (3) cannot both be true. Thus the lemma is proven. □

Note that the lack of a well defined Busemann function implies that there exists at least two disjoint infinite geodesics. Now we show that the lack of a well defined Busemann function also implies coexistence has positive probability. Coexistence is implied if there exist two infinite geodesics $G_0 = (v_0, v_1, v_2, \dots)$ and $G_1 = (w_0, w_1, w_2, \dots)$ such that

$$B_{G_0}(\mathbf{0}, \mathbf{1}) < 0 < B_{G_1}(\mathbf{0}, \mathbf{1}).$$

We show coexistence is possible by showing that we have two such geodesics with positive probability.

Proof of Theorem 2: By Lemma 4 we get an event $\tilde{A}$ of positive probability and $x, y \in \mathbb{Z}^d$ such that for all $\omega \in \tilde{A}$ we have two geodesics $G_0 = G_0(\omega) = (v_0, v_1, v_2, \dots)$ and $G_1 = G_1(\omega) = (w_0, w_1, w_2, \dots)$ with

$$B_{G_0}(x, y) < B_{G_1}(x, y).$$

(If there is more than one pair of geodesics which satisfy this equation we can choose G_0 and G_1 in any measurable manner.) It causes no loss of generality to assume that $|x - y| = 1$. Thus by the symmetry of μ we can assume that $x = \mathbf{0}$ and $y = \mathbf{1}$. As $B_{G_0}(\mathbf{0}, \mathbf{1})$ and $B_{G_1}(\mathbf{0}, \mathbf{1})$ do not depend on any finite number of edges in the geodesics, it causes no loss of generality to assume that $\mathbf{0}, \mathbf{1}$ are not endpoints of any of the edges in G_0 or G_1. By restricting to a smaller event $A \subset \tilde{A}$ of positive probability we get a nonrandom $r > 0$ such that for all $\omega \in A$

$$B_{G_0}(\mathbf{0}, \mathbf{1}) < r < B_{G_1}(\mathbf{0}, \mathbf{1}). \tag{4}$$

By the symmetry of μ we can assume $r \geq 0$. From the definition of $B_{G_1}(\mathbf{0}, \mathbf{1})$ we get that $B_{G_1}(\mathbf{0}, \mathbf{1}) \leq \tau(\mathbf{0}, \mathbf{1})$.

Now we form a new event A'. Given $\omega \in A$ define ω' by

$$\omega'(v, w) = \begin{cases} \omega(v, w) + r, & \text{if } \mathbf{1} \in \{v, w\}; \\ \omega(v, w), & \text{else.} \end{cases}$$

The event A' consists of all ω' that can be formed from in this way from some $\omega \in A$. By conditions 2, 4 and 5 of the definition of μ being good, the event A' also has positive measure. We let τ' indicate the passage times in ω' and τ indicate the passage times in ω. It is easy to check that for any $z \neq \mathbf{1}$

$$\tau'(\mathbf{1}, z) = \tau(\mathbf{1}, z) + r.$$

Also if $\mathbf{1}$ is not an endpoint of any of the edges in the geodesic $G^{\omega}(\mathbf{0}, z)$ then

$$\tau'(\mathbf{0}, z) = \tau(\mathbf{0}, z).$$

As $B_{G_0}(\mathbf{0}, \mathbf{1}) < B_{G_1}(\mathbf{0}, \mathbf{1}) \leq \tau(\mathbf{0}, \mathbf{1})$ we have that for all large n the vertex $\mathbf{1}$ is not an endpoint of any of the edges in the geodesic $G^{\omega}(\mathbf{0}, v_n)$. Thus $\tau'(\mathbf{0}, v_n) = \tau(\mathbf{0}, v_n)$ for all large n. Also note that since neither $\mathbf{0}$ or $\mathbf{1}$ is an endpoint of any of the edges $G_0(\omega)$ or $G_1(\omega)$ we have that $G_0(\omega)$ and $G_1(\omega)$ are both geodesics for ω'.

Thus for any $\omega' \in A'$ we have that

$$\begin{aligned}
B^{\omega'}_{G_0(\omega)}(\mathbf{0}, \mathbf{1}) &= \lim_{n \to \infty} \tau'(\mathbf{0}, v_n) - \tau'(\mathbf{1}, v_n) \\
&= \lim_{n \to \infty} \tau(\mathbf{0}, v_n) - (\tau(\mathbf{1}, v_n) + r) \\
&= B^{\omega}_{G_0(\omega)}(\mathbf{0}, \mathbf{1}) - r \\
&< 0.
\end{aligned}$$

The last step follows from line (4). We also get that

$$\begin{aligned}
B^{\omega'}_{G_1(\omega)}(\mathbf{0}, \mathbf{1}) &= \lim_{n \to \infty} \tau'(\mathbf{0}, w_n) - \tau'(\mathbf{1}, w_n) \\
&\geq \lim_{n \to \infty} \tau(\mathbf{0}, w_n) - (\tau(\mathbf{1}, w_n) + r) \\
&\geq B^{\omega}_{G_1(\omega)}(\mathbf{0}, \mathbf{1}) - r \\
&> 0.
\end{aligned}$$

The last step follows from line (4). Thus we have coexistence for all $\omega' \in A'$.

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□

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