A family of nonisomorphic Markov random fields

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Abstract

It was recently shown that there exists a family $\mathbb{Z}^2$ Markov random fields which are K but are not isomorphic to Bernoulli shifts [4]. In this paper we show that most distinct members of this family are not isomorphic. This implies that there is a two parameter family of $\mathbb{Z}^2$ Markov random fields of the same entropy, no two of which are isomorphic.

1 Introduction

Markov chains are some of the most studied and best understood objects in both probability theory and dynamical systems. Friedman and Ornstein proved that finite state space, mixing Markov chains are isomorphic to Bernoulli shifts [2]. This can be extended to show that any Markov chain is isomorphic to a permutation of a finite set, a Bernoulli shift, or the direct product of these [8]. Mixing Markov shifts satisfy the central limit theorem and have an exponential rate of convergence to their invariant measure.

The theory for $\mathbb{Z}^2$ Markov random fields is much more complicated from both the perspective of probability theory as well as dynamical systems. For example Ledrappier created a simple $\mathbb{Z}^2$ Markov random field which is mixing, but is not 3-mixing, and has zero entropy [5]. Whether there exist actions of $\mathbb{Z}$ which are mixing but not 3-mixing is a long standing open question in ergodic theory [3]. Many other people have expanded upon Ledrappier's example to create a wide variety of zero entropy mixing $\mathbb{Z}^2$ Markov random fields. Even if a $\mathbb{Z}^2$ Markov random field has completely positive entropy (also called K) it need not be isomorphic to a Bernoulli shift [4]. Besides specific examples such as these only a little about $\mathbb{Z}^2$ the ergodic theory behavior of Markov random fields is known.

In this paper we give another example of the differences between the ergodic theoretical properties of Markov random fields and those of Markov chains. We will examine the class of Markov random fields studied in [4]. We will show that in this class that most distinct members are not isomorphic. Specifically we show that there exist an uncountable family of Markov random fields with the same entropy in which no two distinct members are isomorphic.

2 Construction

First we give a heuristic description of a discrete time exclusion processes on the integers which was introduced by Yaguchi [9]. For a rigorous description of this process see [9]. Arrange the integers in a vertical line, and on each integer place a container which can hold at most one particle. At any time the state of our system is given by $x \in X = \{0,1\}^{\mathbb{Z}}$, where $x_i = 1$ implies that there is a particle in the ith container. We first describe the transition probabilities and then describe the stationary measures on X. After one unit of time each of our particles will either stay in the same container or move down one (to a lower numbered) container. To describe the movement of our particles we choose α, $0 < \alpha \leq 1/2$. The process evolves in the following way. During each interval of time, every particle decides independently with probability α if it wants to move down one space. It also checks if the container below it is vacant. If both the particle decides to move, and the container below is empty, then the particle moves down one container. Otherwise it stays in the same container.

The stationary measures on $\{0,1\}^{\mathbb{Z}}$ for this process were classified by Yaguchi. They include a family of point masses. Define $x_m(i) = 0$ for all $i > m$ and $x_m(i) = 1$ for all $i \leq m$. Then for each m the point mass at x_m is stationary. To describe the other stationary measures we define ρ to be the value between 0 and 1 which specifies the density of containers that have a particle. Yaguchi proved that for each α and ρ there is a unique stationary measure on X [9]. Define $\gamma = (1/\rho) - 1$ and

$$\beta = \frac{\gamma + 1 - \sqrt{(\gamma+1)^2 - 4\gamma\alpha}}{2\gamma}.$$

Theorem 2.1 *[9] The nontrivial stationary measures of the exclusion process are the Markovian measures $\mu_{\alpha,\rho}$ given by the transition probabilities*

$$P(0,0) = 1 - \beta/\alpha, P(1,0) = \beta\gamma/\alpha, P(0,1) = \beta/\alpha, P(1,1) = 1 - \beta\gamma/\alpha$$

where $P(p,q) = \mu(x_{i+1} = q | x_i = p)$.

To convert this exclusion process into a two dimensional space-time process we use the space $\Omega = (0,1)^{\mathbb{Z}^2}$. S and T are the shifts down and to the left. That is $(S(\omega))_{i,j} = (\omega)_{i,j+1}$ and $(T(\omega))_{i,j} = (\omega)_{i+1,j}$. Applying T corresponds with time increasing by one unit. The measure $\overline{\mu} = \overline{\mu}_{\alpha,\rho}$ on Ω has measure $\mu = \mu_{\alpha,\rho}$ on vertical cylinder sets and is determined on other cylinder sets by $\mu_{\alpha,\rho}$ and the transition probabilities of the exclusion process.

Theorem 2.2 *[4] For any $\rho \in (0,1)$ and $\alpha \in (0,.5)$ the exclusion process is isomorphic to a Bernoulli shift.*

Proof: This follows easily from calculations in [9]. □

Now we describe how to alter the exclusion processes to form the class of transformations that we will study in this paper. First we choose a value c between 0 and 1. Then we assign to each particle the color red with probability c and blue with probability $1-c$. This is done independently for each particle. The color of a particle does not change through time.

To do this coloring formally we use the exclusion process as a base for a skew product with the one dimensional $(c, 1-c)$ Bernoulli shift. We have already defined $\Omega = \{0,1\}^{\mathbb{Z}^2}$. Now define $Y = \{\text{red, blue}\}^{\mathbb{Z}}$. Define $\sigma : Y \to Y$ by $(\sigma(y))_i = y_{i+1}$. Now the process we are interested in, $(S, T, \Omega \times Y, m)$, is defined as follows.

$$S(\omega, y) = \begin{cases} (S(\omega), \sigma(y)) & \text{if } \omega_{0,0} = 1 \\ (S(\omega), y) & \text{if } \omega_{0,0} = 0 \end{cases}$$

$$T(\omega, y) = \begin{cases} (T(\omega), \sigma(y)) & \text{if } \omega_{0,0} = 1 \text{ and } \omega_{1,0} = 0 \\ (T(\omega), y) & \text{else} \end{cases}$$

S and T are the shifts down and to the left and σ is the left shift on Y. The measure $m = \overline{\mu} \times \nu$, where the measure ν on Y is $(c, 1-c)$ product measure. Ocassionally we will work with the measure $\hat{m} = \overline{\mu} \times \nu \times \nu$ on the space $\Omega \times Y \times Y$. With the two shifts this measure represents an exclusion process where the particles may have four colors.

We will work with the three set partition, P, which takes on values 0, red, and blue. This partition tells us whether there is a ball in container 0 at time 0 and, if so, what color it is. It is easy to check that these colored exclusion process are Markov random fields. The main result from [4] tells us that all of these systems are Markov random fields which are K but not isomorphic to Bernoulli shifts.

Theorem 2.3 [4] *For any choice of $\alpha \in (0,.5), \rho \in (0,1)$, and $c \in (0,1)$ the resulting transformation is a Markov random field which is K but is not isomorphic to a Bernoulli shift.*

There is one factor of the colored exclusion process that will play a large role in the proof. That is the factor generated by the factor map $F(\omega, y) = \omega$. We refer to this as the **exclusion process factor**. This factor is isomorphic to a Bernoulli shift by Theorem 2.2.

The main result of this paper is that almost any two choices of α, ρ, and c result in transformations which are not isomorphic. We will state the theorem precisely in the next section after we have introduced some more notation.

3 Main Theorem

In this section we state our main results. In the construction of the family of colored exclusion processes there are three degrees of freedom. They are α, the propensity to move downward, ρ, the density of particles, and c the percentage of particles colored blue. During the proof, however, it will be advantageous to use a different set of three parameters. We call these parameters the entropy of the colored exclusion process (as a $\mathbb{Z}^2$ action), the average speed of a particle, and the entropy of the colored exclusion process relative to the exclusion process under the action of S. The formula for the entropy of the colored exclusion process is

$$H(\alpha, \rho, c) = h(\alpha)\mu_{\alpha,\rho}(x_0 = 0, x_1 = 1) = \frac{1 - \sqrt{1 - 4\alpha(\rho - \rho^2)}}{2\alpha} h(\alpha).$$

(We are using the notation $h(c) = -(c \log(c) + (1 - c)\log(1 - c))$.) The average speed of a particle is given by the formula

$$s(\alpha, \rho, c) = \alpha\mu_{\alpha,\rho}(x_0 = 0, x_1 = 1)/\rho = \frac{1 - \sqrt{1 - 4\alpha(\rho - \rho^2)}}{2\rho}.$$

(We will explain where this formula comes from in the next section.) The relative entropy is given by the formula

$$r(\alpha, \rho, c) = \rho h(c).$$

The main result of this paper is to show that for most triples (α, ρ, c) and (α', ρ', c') the colored exclusion processes generated by these triples are not isomorphic. If the two colored exclusion processes are isomorphic then they must have the same entropy (i.e. $H(\alpha, \rho, c) = H(\alpha', \rho', c')$). Most of this paper is building up to the proof of Lemma 5.1. From this it will be easy to show that if the two colored exclusion processes are isomorphic then they have the same speed (i.e. $s(\alpha, \rho, c) = s(\alpha', \rho', c')$). With more work we will show that Lemma 5.1 implies that if the two colored exclusion processes are isomorphic then $r(\alpha, \rho, c) = r(\alpha', \rho', c')$. These results combine to form our main theorem.

Theorem 3.1 *If the colored exclusion process given by* (α, ρ, c) *is isomorphic to the colored exclusion process given by* (α', ρ', c') *then* $H(\alpha, \rho, c) = H(\alpha', \rho', c')$*,* $s(\alpha, \rho, c) = s(\alpha', \rho', c')$*, and* $r(\alpha, \rho, c) = r(\alpha', \rho', c')$*.*

By looking at the partial derivatives of the function from α and ρ to H and s it is not difficult to see that it is at most two to one. In the region where α and ρ are less than one half it is one to one. This gives us the following corollary.

Corollary 3.1 *If* $0 < \alpha, \alpha, \rho, \rho', c, c' \leq .5$ *then the colored exclusion process generated by* (α, ρ, c) *is isomorphic to the colored exclusion process generated by* (α', ρ', c') *if and only if* $\alpha = \alpha'$*,* $\rho = \rho'$ *and* $c = c'$*.*

Proof: By checking the partial derivatives it is easy to see that the map from α and ρ to H and s is one to one in this region. (In the interior of this region $H_\alpha > 0$, $H_\rho > 0$, $s_\alpha > 0$ and $s_\rho < 0$ which justifies the claim.) Thus the previous theorem implies that $\alpha = \alpha'$, and $\rho = \rho'$. As $h(c)$ is increasing in this region the fact that $\rho = \rho'$ implies $c = c'$. □

Corollary 3.2 *There exists a two parameter family of* $\mathbb{Z}^2$ *Markov random field which are all* K *and have the same entropy but no two are isomorphic.*

4 Notation

In this section we present much of the notation used in the proof. For the rest of the paper we fix (α, ρ, c) and (α', ρ', c') and assume that Φ is an isomorphism between the colored exclusion proccesses generated by (α, ρ, c) and (α', ρ', c'). We have already defined the exclusion process factor by the projection map $F(\omega, y) = \omega$. We will consider the other projection map $C(\omega, y) = y$. We use this projection to define $\Phi^*(\omega, y) = C(\Phi(\omega, y))$.

Given $\omega \in \Omega$ we number the particles. Let $I_\omega = \{j : \omega_{0,j} = 1\}$, the set of containers which have particles at time 0. For any $k \geq 0$ define ω^k to be the smallest $j \geq 0$ such that $|I_\omega \cap [0, j]| = k + 1$. For $k < 0$ let ω^k be the largest j such that $|I_\omega \cap [j, -1]| = -k$. Particle i is defined to be the particle in container ω^i. For $i \geq 0$ this is the $i+1$st particle you see when you start at container 0 and then move upward. Let $\omega^{i,t}$ be the number of the container that particle i is in at time t. To be more precise set $\omega^{i,0} = \omega^i$. Given $\omega^{i,t}$ we set $\omega^{i,t+1} = \omega^{i,t} - 1$ if $\omega_{t+1,\omega^{i,t}} = 0$ and $\omega^{i,t+1} = \omega^{i,t}$ otherwise.

At various times during the proof we will use properties of a **free range process**. This process is closely related to the exclusion process. This process acts on the space $R = \{\mathbb{N}\}^{\mathbb{Z}^2}$. Define the set $\Omega_1 = \{\omega \in (0,1)^{\mathbb{Z}^2} : \omega_{0,0} = 1\}$. Define a map $f : \Omega_1 \to R$ by

$(f(\omega))_{i,t} = \omega^{i+1,t} - \omega^{i,t}$. We put a measure n on R by $n(A) = \bar{\mu}(\{\omega : \ \omega \in \Omega_1, f(\omega) \in A\}$. The shift operator σ acts on R by $\sigma(x)_{i,j} = x_{i+1,j}$.

Theorem 4.1 *The process (R, σ, n) is isomorphic to an infinite entropy Bernoulli shift.*

Proof: This follows easily from arguments in [4]. □

We will only use that the free range process is ergodic in this paper.

By the average speed of particle 0 in ω we mean $s = -\lim_{t\to\infty} \omega^{0,t}/t$. By the ergodicity of the free range process this limit is the same for almost every ω. The connection between the exclusion process and the free range process tells us that the average speed is also

$$\bar{\mu}(\{\omega : \ \omega_{0,0} = 1 \text{ and } \omega_{1,0} = 0\}),$$

which agrees with the formula in the previous section.

During the course of the proof we will use the $\bar{d}$ and $\bar{f}$ metrics often. The $\bar{d}$ and $\bar{f}$ metrics are defined as follows. For any m, n, and $w, w' \in \{\text{red,blue}\}^{\mathbb{Z}}$ let

$$\bar{d}_{[m,n]}(w, w') = \frac{\{j : w(j) \neq w'(j),\ m \leq j \leq n\}}{n - m + 1},$$

For any m, n, and $w, w' \in \{\text{red,blue}\}^{\mathbb{Z}}$ let

$$\bar{f}_{[m,n]}(w, w') = 1 - \frac{k}{n - m + 1},$$

where k is the maximal integer for which there are subsequences of integers, $m \leq i_1 < i_2 < \ldots < i_k \leq n$ and $m \leq j_1 < j_2 < \ldots < j_k \leq n$ such that $w(i_r) = w'(i_r)$, $1 \leq r \leq k$. An ϵ $\bar{f}_{[0,n]}$ neighborhood of w is the set $\{w' : \ f_{[0,n]}(w, w') < \epsilon\}$. Simple combinatorics show that for any c, $0 < c < 1$, there is a sufficiently small ϵ such that for all w the $c, 1-c$ product measure of the ϵ $\bar{f}_n$ neighborhood of w is decreasing exponentially in n.

Our proof will be based on studying finite code approximations to isomorphisms. Since Φ is measurable we can choose an n large enough so that there exists Φ_n which is an ϵ good finite approximation of Φ. That means Φ_n is a function $\Phi_n : \Omega \times Y \to P$ such that

1. $m\left(\{(\omega, y) : \ \Phi_n(\omega, y) \neq \Phi(\omega, y)_{0,0}\}\right) < \epsilon$ and

2. $\Phi_n(\omega, y) = \Phi_n(\omega', y')$ if $(\omega, y)_{i,j} = (\omega', y')_{i,j}$ for all i, j, such that $-n < i, j < n$.

Now we describe the main way which we use the finite codes. The sequence $\Phi_n(S^j(\omega, y))$ approximates the sequence $\Phi(\omega, y)_{0,j}$. We would like to find a sequence that approximates $\Phi^*(\omega, y)_j$. The sequence $\Phi(\omega, y)_{0,j}$ has $\Phi^*(\omega, y)_j$ as a subsequence. The only symbols in $\Phi(\omega, y)_{0,j}$ but not $\Phi^*(\omega, y)_j$ are all 0. The procedure defined below lets us extract the subsequence $\Phi^*(\omega,y)_j$ from the sequence $\Phi(\omega,y)_{0,j}$. Now we define how we **remove the zeroes** from a sequence $a_j \in \{0, \text{ red, blue}\}^{\mathbb{Z}}$ to get a sequence $\bar{a}_j \in \{\text{red, blue}\}^{\mathbb{Z}}$. Let $I = \{j | \, a_j \in \{\text{red, blue}\}\}$. For $k \geq 0$ define e_k to be the smallest j such that $|I \cap [0,j]| = k+1$. For $k < 0$ define e_k to be the largest j such that $|I \cap [j,-1]| = -k$. The sequence we get from removing the zeros from a_j is $\bar{a}_j = a_{e_j}$.

The reason we made this definition is the following lemma. It tells us that if two sequences are close in $\bar{d}$ then when we remove the zeroes we get two sequences that are close in $\bar{f}$.

Lemma 4.1 *If* $a, b \in \{0, \textit{ red, blue}\}^{\mathbb{N}}$, $\lim_{J\to\infty} \#\{j \in [0,J] : a_j \neq 0\}/(J+1) = \rho$, *and* $\lim_{J'\to\infty} \#\{j \in [0,J'] : a_j \neq b_j\}/(J'+1) < \epsilon$, *then* $\lim_{k\to\infty} \bar{f}_{[0,k]}(\bar{a},\bar{b}) < \epsilon/\rho$.

Proof: Let $I_a = \{j : a_j \in \{\text{red, blue}\}\}$. Let $I_b = \{j : b_j \in \{\text{red, blue}\}\}$. Let $I = \{j : a_j = b_j \in \{\text{red, blue}\}\}$. I is a subset of both I_a and I_b and has density at least $\rho - \epsilon$. Thus there is a subsequence of density greater than $(\rho-\epsilon)/\rho$ in $\bar{a}$ that is the same as a subsequence in $\bar{b}$. Thus $\lim_{k\to\infty} \bar{f}_{[0,k]}(\bar{a},\bar{b}) < \epsilon/\rho$. □

Define $\Phi^*_n(\omega,y)$ to be the sequence obtained from removing the zeros from $\Phi_n(S^j(\omega,y))$. The sequence $\Phi^*_n(\omega,y)$ approximates $\Phi^*(\omega,y)$. At each time t we also construct the sequence (which is indexed by j) $\Phi_n(T^t S^{j+(F(\Phi(\omega,y))^{0,t})}(\omega,y))$. Define $\bar{\Phi}^t_n(\omega,y)$ to be the sequence obtained from eliminating the zeros from the previous sequence. For each t the sequence $\bar{\Phi}^t_n(\omega,y)$ approximates $\Phi^*(\omega,y)$.

For each time t we construct the sequence $\Phi_n(T^t S^{j+\omega^{0,t}}(\omega,y))$. We eliminate the zeros from this sequence to get the new sequence $\Phi^{*,t}_n(\omega,y)$. Notice that the two sequences $\bar{\Phi}^t_n(\omega,y)$ and $\Phi^{*,t}_n(\omega,y)$ are translates of each other. We now use this fact to make one more definition. This will play a crucial role in Section 5. Define $d^t_n = d^t_n((\omega,y),\Phi_n)$ so that

$$\bar{\Phi}^t_n(\omega,y) = \sigma^{-d^t_n}(\Phi^{*,t}_n(\omega,y)).$$

5 The main lemma

Remember we have fixed (α,ρ,c) and (α',ρ',c'). We have assumed Φ is an isomorphism between the colored exclusion processes generated by (α,ρ,c) and (α',ρ',c'). We have Φ_n a sequence of finite approximations which converge to Φ. We also have that $\Phi^*_n(\omega,y)$ and $\bar{\Phi}^t_n(\omega,y)$ are sequences that approximate $\Phi^*(\omega,y)$ and $\Phi^{*,t}_n(\omega,y)$ is a sequence that approximates $\sigma^{d^t_n}(\Phi^*(\omega,y))$. Our goal in this section is to prove the following lemma.

Lemma 5.1 *There exists a* Φ_n, N, *and an* $\epsilon > 0$ *such that for all* T

$$m(\{(\omega,y) : \ \# \, \{t \in [0,T] \text{ such that } |d^t_n(\omega,y)| < N\} \ < \epsilon(T+1)\}) \leq 1-\epsilon.$$

An outline of the proof is as follows. Given y and y' pick z so that $z_i = y_i$ for all $i \geq 0$ and $z_i = y'_i$ for all $i < 0$. We will first show that

$$\lim_{k\to\infty} \bar{f}_{[0,k]}(\Phi^*(\omega, y), \Phi^*(\omega, z)) = 0.$$

Then we will show in Lemmas 5.5 and 5.6 that no matter how big t becomes

$$|d_n^t(\omega, y) - d_n^t(\omega, z)|$$

usually stays bounded. We will combine this with coding arguments and the assumption that Lemma 5.1 is false to conclude that

$$\liminf_{k\to\infty} \bar{f}_{[-k,k]}(\Phi^*(\omega, y), \Phi^*(\omega, z)) = 0.$$

Similar arguments show that

$$\liminf_{k\to\infty} \bar{f}_{[-k,k]}(\Phi^*(\omega, y'), \Phi^*(\omega, z)) = 0.$$

We show that the lim infs can be achieved on a subsequence of density near 1. Thus if Lemma 5.1 is false then

$$\liminf_{k\to\infty} \bar{f}_{[-k,k]}(\Phi^*(\omega, y), \Phi^*(\omega, y')) = 0.$$

This statement is Lemma 5.7 and forms the heart of the proof of Lemma 5.1. Then we complete the proof of Lemma 5.1 by using a variant of Lemma 5.7 and Theorem 2.2 to generate a contradiction.

Lemma 5.2 *For any $\epsilon > 0$ there exists Φ_n such that for almost every (ω, y)*

$$\lim_{k\to\infty} \bar{f}_{[0,k]}(\Phi_n^*(\omega, y), \Phi^*(\omega, y)) < \epsilon.$$

Proof: Let Φ_n be a finite approximation of Φ which is $\rho'\epsilon$ good. We apply the finite code Φ_n to the sequence of points $S^j(\omega, y)$ for all $j > 0$. The colored exclusion process is totally ergodic, so by the ergodic theorem for almost every (ω, y) the density of j such that $\Phi_n(S^j(\omega, y)) \neq \Phi(\omega, y)_{0,j}$ is less than or equal to $\rho'\epsilon$. The ergodic theorem also tells us that for almost every (ω, y) the density of particles in $\Phi(\omega, y)$ is ρ'. By Lemma 4.1 when we remove the zeros we get that

$$\lim_{k\to\infty} \bar{f}_{[0,k]}(\Phi_n^*(\omega, y), \Phi^*(\omega, y)) < \epsilon.$$

□

Lemma 5.3 *For almost every pair (ω, y) and (ω, z) such that $z_i = y_i$ for all $i \geq 0$*

$$\lim_{k\to\infty} \bar{f}_{[0,k]}(\Phi^*(\omega, y), \Phi^*(\omega, z)) = 0.$$

Proof: Since $z_i = y_i$ for all $i \geq 0$ we have that $\Phi_n^*(\omega, y)_j = \Phi_n^*(\omega, z)_j$ for all $j > 2n$. Thus Lemma 5.2 implies that $\lim_{k\to\infty} \bar{f}_{[0,k]}(\Phi^*(\omega, y), \Phi^*(\omega, z)) \leq 2\epsilon$. As ϵ is arbitrary the proof is complete. □

Define

$$B_1 = B_1((\omega, y), J_0, \epsilon) = \{t : \text{ there exists } J > J_0 \text{ such that } \bar{f}_{[0,J]}(\sigma^{d_n^t}(\Phi^*(\omega, y), \Phi_n^{*,t}(\omega, y))) > \epsilon,$$

$$B_2 = B_2((\omega, y), J_0, \epsilon) = \{t : \text{ there exists } J > J_0 \text{ such that } \bar{f}_{[0,J]}(\bar{\Phi}_n^t(\omega, y), \Phi^*(\omega, y)) > \epsilon, \tag{1}$$

and

$$B_3 = B_3((\omega, y), J_0, \epsilon) = \{t : \text{ there exists } J > J_0 \text{ such that } \bar{f}_{[-J,0]}(\Phi^*(\omega, y), \bar{\Phi}_n^t(\omega, y)) > \epsilon\}.$$

Then set $B = B_1 \cup B_2 \cup B_3$.

Lemma 5.4 *For any $\epsilon, \epsilon_0 > 0$ there exists Φ_n and J_0 so that for all T*

$$m\{(\omega, y) : \#\{B \cap [0, T]\} < \epsilon(T + 1)\} > 1 - \epsilon_0.$$

Proof: For each k and t such that $S^k(\omega) \in \Omega_1$ and $t \in B_2(S^k(\omega, y), J_0, \epsilon)$ there exists a corresponding j and $J > J_0$ such that

$$\bar{f}_{[0,j]}(\Phi_n^*(S^j T^t(\omega, y)), \Phi^*(S^j T^t(\omega, y))) > \epsilon\}. \tag{2}$$

The correspondence between k, t and j, t is one to one. By the ergodic theorem the density of $j \in \mathbb{Z}$ and $0 \leq t \leq T$ such that line 2 is not satisfied can be made less than any $\epsilon' > 0$ by Lemma 5.2. By the ergodic theorem

$$\int \#\{B \cap [0, T]\}$$

is the density of $k \in \mathbb{Z}$ and $0 \leq t \leq T$ such that $S^k(\omega) \in \Omega_1$ and line 1 is not true. By the correspondence above this density is bounded by ϵ'/rho'. Thus by Fubini's theorem we have that for all T

$$\int \#\{B_2 \cap [0, T]\}\ dm \leq \epsilon'/\rho'.$$

Similar arguments work for B_1 and B_3. Thus

$$\int \#\{B \cap [0, T]\}\ dm \leq \epsilon'/\rho'.$$

As ϵ is arbitrary the right hand side can be made arbitrarily small. Thus the lemma is true. $\square$

Our next goal is to show that if Lemma 5.1 is false then

$$\liminf_{k\to\infty} \bar{f}_{[-k,0]}(\Phi^*(\omega,y),\Phi^*(\omega,z)) = 0.$$

The basic idea is that if $d_n^t = -N$ then $\Phi_n^{*,t}(\omega,y)_i$ for $0 \le i \le N$ approximates $\Phi^*(\omega,y)_i$ for $-N \le i \le 0$. The first step is to show that by comparing the sequences $\Phi_n^{*,t}(\omega,y)$ and $\Phi_n^*(\omega,y)$ we can determine the value of d_n^t fairly well no matter how big t becomes. Moreover we will show that if $y_i = z_i$ for all $i \ge 0$ then $|d_n^t(\omega,y) - d_n^t(\omega,z)|$ is bounded on a set of t of high density. This will be enough to be able to use coding arguments to draw the desired conclusion.

Given $\epsilon > 0$ define $C_{J_0} \subset Y$ by

$$C_{J_0} = \left\{y :\ \text{there exists a } J > J_0 \text{ such that } \bar{f}_{[0,J]}(\sigma^J y, y) < \epsilon\right\}.$$

Because the ν' measure of an ϵ $\bar{f}_{[0,k]}$ neighborhood decreases exponentially in k for any sufficiently large J_0 we have

$$\nu'(C_{J_0}) < \epsilon. \tag{3}$$

Consider the set

$$L = L(t,\epsilon,(\omega,y)) = \{l :\ \bar{f}_{[0,J]}(\Phi_n^*(\omega,y), \sigma^{-l}(\Phi_n^{*,t}(\omega,y))) < \epsilon/10 \text{ for all } J > J_0.\}$$

If (ω,y) satisfies

$$\bar{f}_{[0,J]}(\Phi_n^*(\omega,y),\Phi^*(\omega,y)) < \epsilon/20 \tag{4}$$

for all $J > J_0$, and for a given t and all $J > J_0$

$$\bar{f}_{[0,J]}(\bar{\Phi}_n^t(\omega,y),\Phi^*(\omega,y)) < \epsilon/20. \tag{5}$$

then $d_n^t \in L$. This is because $\bar{\Phi}_n^t(\omega,y) = \sigma^{-d_n^t}(\Phi_n^{*,t}(\omega,y))$. Thus L is the set of possible values of d_n^t consistent with $\Phi_n^*(\omega,y)$ and $\Phi_n^{*,t}(\omega,y)$ given good coding. We will show that there exists a bound on the diameter of L which holds for most (ω,y) and t.

Lemma 5.5 *If (ω,y) satisfies line 4 for all $J > J_0$ and diam$(L) \ge J_0$ for some t then $\Phi^*(\omega,y) \in C_{J_0}$.*

Proof: Let (ω,y) satisfy line 4 and let $l_1,l_2 \in L$ such that $J' = l_2 - l_1 > J_0$. Then

$$\bar{f}_{[0,J']}(\Phi_n^*(\omega,y),\sigma^{-l_1}(\Phi_n^{*,t}(\omega,y))) < \epsilon/10 \tag{6}$$

and

$$\bar{f}_{[0,J']}(\sigma^{J'}(\Phi_n^*(\omega,y)),\sigma^{-l_1}(\Phi_n^{*,t}(\omega,y))) < 2\epsilon/10. \tag{7}$$

Line 6 is true because $l_1 \in L$ and by using $J = J'$. Line 7 is true because $l_2 \in L$ and by using $J = 2J'$. The triangle inequality implies that

$$\bar{f}_{[0,J']}(\Phi_n^*(\omega,y),\sigma^{J'}(\Phi_n^*(\omega,y))) < 3\epsilon/10.$$

Line 4 with $J = J'$ and $J = 2J'$ and the triangle inequality imply that

$$\bar{f}_{[0,J']}(\Phi^*(\omega,y),\sigma^{J'}(\Phi^*(\omega,y))) < 6\epsilon/10.$$

So $\Phi^*(\omega,y) \in C_{J_0}$. □

We will use L to define a function $F(\omega,y,t) \to \mathbb{Z}$ so that $F(\omega,y,t) = F(\omega,z,t)$ if $y_i = z_i$ for all $i \geq 0$ and $F(\omega,y,t)$ approximates $d_n^t(\omega,y)$. To measure how well F approximates $d_n^t(\omega,y)$ we define

$$G_{a,\epsilon,T,F} = \{(\omega,y) |\ \#\ \{t \in [0,T]:\ |F(\omega,y,t) - d_n^t(\omega,y)| < a\} > (1-\epsilon)(T+1)\}.$$

If $m()$ is near one then for most points (ω,y) and most times $t \leq T$ the quantities $d_n^t(\omega,y)$ and $F(\omega,y,t)$ by less than a.

Lemma 5.6 *For any $\epsilon > 0$ there exists a function F such that $F(\omega,y,t) = F(\omega,z,t)$ if $y_i = z_i$ for all $i \geq 0$ and*

$$\lim_{a\to\infty} \inf_T m(G_{a,\epsilon,T,F}) = 1.$$

Proof: J_0 will be specified later. We define F as follows. Given any t, ω, and $a_i, i \geq 0$, find y such that

1. $a_i = y_i$ for all $i \geq 0$,
2. $\bar{f}_{[0,J]}(\Phi_n^*(\omega,y),\Phi^*(\omega,y)) < \epsilon/20$ for all $J \geq J_0$,
3. $\bar{f}_{[0,J]}(\bar{\Phi}_n^t(\omega,y),\Phi^*(\omega,y)) < \epsilon/20$ for all $J \geq J_0$, and
4. $\Phi^*(\omega,y) \notin C_{J_0}$.

Then define $F(\omega,y,t)$ to be any $l \in L$. If (ω,z) and t also satisfies conditions 1 through 4 then by the previous two lemmas $d_n^t(\omega,z) \in L(\omega,y,t)$ and

$$|F(\omega,z,t) - d_n^t(\omega,z)| < J_0.$$

We need to show that the measure of $G_{J_0,\epsilon,T,F}$ can be made arbitrarily close to 1. If (ω, z) satisfies conditions 2 and 4 and

$$\#\{t: \ t \in [0,T] \text{ and } \bar{f}_{[0,J]}(\bar{\Phi}^t_n(\omega,z), \Phi^*(\omega,z)) < \epsilon/20 \text{ for all } J > J_0\} > (1-\epsilon)(T+1) \quad (8)$$

then $(\omega, z) \in G_{J_0,\epsilon,T,F}$. Lemma 5.2 shows that for large J_0 the measure of the set of (ω, z) that don't satisfy condition 2 is less than ϵ. By Lemma 5.4 the measure of points that don't satisfy the line 8 can also be made less than ϵ for large J_0. We also have that $\nu'(C_{J_0}) < \epsilon$ for large J_0.

Thus for any sufficiently large J_0 and any T we have $m(G_{J_0,\epsilon,T,F}) > 1 - 3\epsilon$. As ϵ was arbitrary

$$\lim_{a\to\infty} \inf_T m(G_{a,\epsilon,T,F}) = 1.$$

□

Lemma 5.7 *If Lemma 5.1 is not true then for almost all* (ω, y) *and* (ω, y')

$$\liminf_{k\to\infty} \bar{f}_{[-k,k]}(\Phi^*(\omega,y), \Phi^*(\omega,y')) = 0.$$

Proof: If Lemma 5.1 is not true then for any N, T, and sufficiently small ϵ there exists Φ_n so that either

$$m(\{(\omega,y): \ \# \ \{ \ t \in [0,T] \text{ such that } d^t_n < -N\} \ < 5\epsilon(T+1)\}) \leq 1 - 5\epsilon, \quad (9)$$

or

$$m(\{(\omega,y): \ \# \ \{ \ t \in [0,T] \text{ such that } d^t_n > N\} \ < 5\epsilon(T+1)\}) \leq 1 - 5\epsilon. \quad (10)$$

Without loss of generality we will assume that it is the former. If both are true then we could replace the lim infs with a limit.

Fix $\epsilon < 1/100$. For the moment assume that N, n, and Φ_n are fixed. We want a to be large enough so that for all T sufficiently large

$$m(G_{a,\epsilon,T,F}) > 1 - \epsilon. \quad (11)$$

This can be satisfied because of Lemma 5.6. Given y and y' let z be chosen so that $z_i = y_i$ for all $i \geq 0$ and $z_i = y'_i$ for all $i < 0$.

Let D be the set of three-tuples (ω, y, y') such that there exists a t where

1. $F(\omega, y, t) = F(\omega, z, t) < -N$,

2. $|F(\omega, y, t) - d_n^t(\omega, y)| < a$,

3. $|F(\omega, z, t) - d_n^t(\omega, z)| < a$,

4. $\bar{f}_{[-k,0]}(\bar{\Phi}_n^t(\omega, y), \Phi^*(\omega, y)) < \epsilon$ for all $k \geq a$, and

5. $\bar{f}_{[-k,0]}(\bar{\Phi}_n^t(\omega, z), \Phi^*(\omega, z)) < \epsilon$ for all $k \geq a$.

By lines 9 and 11 and Lemma 5.4 it is possible to choose a, n, and Φ_n and arbitrarily large N so that

$$\hat{m}(D) > 0.$$

Because $y_i = z_i$ for all $i \geq 0$

$$\Phi_n^{*,t}(\omega, y)_i = \Phi_n^{*,t}(\omega, z)_i$$

for all $i \geq 2n$. Combined with the facts that $|d_n^t(\omega, y) - d_n^t(\omega, z)| < 2a$,

$$\bar{\Phi}_n^t(\omega, y) = \sigma^{-d_n^t(\omega,y)}(\Phi_n^{*,t}(\omega, y))$$

and

$$\bar{\Phi}_n^t(\omega, z) = \sigma^{-d_n^t(\omega,z)}(\Phi_n^{*,t}(\omega, z))$$

there exists a j, $|j| < 2a$, such that

$$\bar{\Phi}_n^t(\omega, y)_i = \bar{\Phi}_n^t(\omega, z)_{i+j}$$

for all $i > d_n^t(\omega, y) + 2n$. In conjunction with conditions 4 and 5 this implies that

$$\bar{f}_{[-k,0]}(\Phi^*(\omega, y), \Phi^*(\omega, z)) < 2\epsilon + 2a/k$$

for all k such that $J_0 < k < N - 2n$. As ϵ is arbitrary, and for a fixed ϵ and a the choice of N is arbitrary we get that

$$\lim_{k\to\infty} \bar{f}_{[-k,0]}(\Phi^*(\omega, y), \Phi^*(\omega, z)) = 0.$$

By Lemma 5.2

$$\lim_{k\to\infty} \bar{f}_{[-k,0]}(\Phi^*(\omega, y'), \Phi^*(\omega, z)) = 0.$$

Thus

$$\lim_{k\to\infty} \bar{f}_{[-k,0]}(\Phi^*(\omega, y), \Phi^*(\omega, y')) = 0. \tag{12}$$

The set of triples (ω, y, y') which satisfy line 12 is shift invariant. It has positive $(\hat{m})$ measure because it contains D which has positive measure. Thus by the ergodic theorem it has $(\hat{m})$

measure 1. By choosing z so that $z_i = y_i$ for all $i \geq k$ and $z_i = y'_i$ for all $i < k$ and repeating the previous analysis we can show that for most ω, y, and y'

$$\bar{f}_{[-k,k]}(\Phi^*(\omega, y), \Phi^*(\omega, y')) < \epsilon. \tag{13}$$

As ϵ and k were arbitrary this implies that

$$\liminf_{k\to\infty} \bar{f}_{[-k,k]}(\Phi^*(\omega, y), \Phi^*(\omega, y')) = 0.$$

□

We will use the previous lemma in a slightly different form.

Lemma 5.8 *If Lemma 5.1 is false then for any $\epsilon > 0$ there exists an n large enough so that there exists a J_0 and a function $M : \Omega \times Y \to \{red,\ blue\}^{\mathbb{Z}}$ so that*

1. *for any N if $\omega_{i,j} = \omega'_{i,j}$ for all but m pairs (i, j) such that $|i| < n$ and $|j| < N$ then*

$$\#\left\{l \in [0, N] :\ M(\omega, y)_l \neq M(\omega', y')_l\right\} \leq m(2n + 1)$$

and

2. *$m(\{(\omega, y) :\ \bar{f}_{[0,J]}(M(\omega, y), \Phi^*(\omega, y)) < \epsilon\}) > 1 - \epsilon$ for all $J > J_0$.*

Proof: Line 13 shows the existance of n and a map $M' : \Omega \times Y \to \{\text{red, blue}\}^{[-k,k]}$ such that $M'(\omega, y) = M'(\omega', y')$ if $(\omega, y)_{i,j} = (\omega', y')_{i,j}$ for all $|i|, |j| < n$. M' also has the property that

$$m(\{(\omega, y) :\ \bar{f}_{[-k,k]}(M'(\omega, y), \Phi^*(\omega, y)) > \epsilon/10\}) < \epsilon^2/10.$$

Concatenating M' gives the desired M. □

Given $\epsilon > 0$, N, and M from Lemma 5.8 we say a point (ω, y) **codes well for** M if

1. $\bar{f}_{[0,N]}(M(T^{-n}(\omega, y)), \Phi^*(T^{-n}(\omega, y))) < \epsilon$ and
2. $\bar{f}_{[0,N]}(M(T^{i}(\omega, y)), \Phi^*(T^{i}(\omega, y))) < \epsilon$ for at least $3\epsilon\rho' N/4$ of the $i \in (0, \epsilon\rho' N)$.

Lemma 5.8 implies if $N > J_0$

$$m(\{(\omega, y) :\ (\omega, y) \text{ codes well for } M\}) > 1 - 5\epsilon. \tag{14}$$

Let Q be the partition of $\Omega \times Y$ determined by $\omega_{0,0}$. Let

$$Q_{\omega,N} = \left\{\omega' :\ \omega_{i,j} = \omega'_{i,j} \text{ for all } (i, j) \text{ such that } i < 0 \text{ or } i < N \text{ and } j < 0\right\}.$$

The set $Q_{\omega,N}$ and Φ^* induces a measure $\nu_{\omega,N}$ on $\{\text{red, blue}\}^{\mathbb{Z}}$ by

$$\nu_{\omega,N}(A) = (m|_{Q_{\omega,N}})(\{(\omega', y') : \Phi^*(\omega', y') \in A\}).$$

We say a measure λ is ϵ **contained in an** ϵ $\bar{f}_{[0,N]}$ **neighborhood** is there exists a set S such that $\lambda(S) > 1 - \epsilon$ and $\bar{f}_{[0,N]}(a, b) < \epsilon$ for each $a, b \in S$.

Lemma 5.9 *If Lemma 5.1 is not true then for any $\epsilon > 0$ there exists N_0 such that for all $N > N_0$*

$$\mu(\{\omega : \nu_{\omega,N} \text{ is } \epsilon \text{ contained in an } \epsilon\ \bar{f}_{[0,N]} \text{ neighborhood}\}) > 1 - \epsilon. \tag{15}$$

Proof: Assume Lemma 5.1 is not true. Then the conclusion of Lemma 5.8 is true with ϵ replaced by $\epsilon^2/20$. Let M, $J_0 = N_0$ and n be from Lemma 5.8. Let $N_0 > 4n/\epsilon$. Thus by line 14 for all $N > J_0$ the set of points that don't code well for M has measure at most $\epsilon^2/4 + N_0$. For any $\omega', \omega'' \in Q_{\omega,N}$

$$M(T^{-n}(\omega', y')) = M(T^{-n}(\omega'', y'')).$$

Thus if $\omega', \omega'' \in Q_{\omega,N}$ and both (ω', y') and (ω'', y'') code well for M then condition 1 in the definition of codes well for M implies

$$\bar{f}_{[0,N]}(\Phi^*(T^{-n}(\omega', y')), \Phi^*(T^{-n}(\omega'', y''))) < \epsilon^2/10.$$

Because $N > 4n/\epsilon$

$$\bar{f}_{[0,N]}(\Phi^*(\omega', y'), \Phi^*(T^{-n}(\omega', y'))) < \epsilon/4$$

and

$$\bar{f}_{[0,N]}(\Phi^*(\omega'', y''), \Phi^*(T^{-n}(\omega'', y''))) < \epsilon/4.$$

Thus

$$\bar{f}_{[0,N]}(\Phi^*(\omega', y'), \Phi^*(\omega'', y'')) < \epsilon.$$

Thus if all but an ϵ fraction of the points in $Q_{\omega,N}$ code well for M then $\nu_{(\omega,y),N}$ is ϵ contained in an ϵ $\bar{f}_{[0,N]}$ neighborhood. By Fubini's theorem this proves the lemma. □

Now we show that if Lemma 5.1 is false then Lemma 5.8 and Theorem 2.2 imply for most ω the measure $\nu_{\omega,N}$ is not ϵ contained in an ϵ $\bar{f}_{[0,N]}$ neighborhood. Theorem 2.3 and the very weak Bernoulli condition for $\mathbb{Z}^2$ actions give us the following lemma.

Lemma 5.10 *As the factor generated by Q is isomorphic to a Bernoulli shift for every $r > 0$ and $\epsilon > 0$ and any arbitrarily large N and a set G of measure at least $1 - \epsilon$ so that for any point $(\omega, y) \in G$*

$$\bar{d}^{Q}_{[0,rN]\times[0,N]}(m, (m|_{Q_{\omega,N}})) < \epsilon. \tag{16}$$

Proof: This is just a restatement of the very weak Bernoulli condition for the factor generated by Q [1]. □

Lemma 5.11 *If Lemma 5.1 is not true then for any $\epsilon > 0$ there exists an N_0 such that for all $N \geq N_0$, there exists a set S such that $m(S) > 1-\epsilon$ with the following property. For any $(\omega, y) \in S$ there exists a coupling c of ν' and $\nu_{\omega,N}$ with*

$$c(\{(y_1, y_2) : \ \bar{f}_{[0,N]}(y_1, y_2) < 5\epsilon\}) > .2. \tag{17}$$

Proof: Choose N_0 as follows. First use Lemma 5.8 to get M, J_0 and n. By Theorem 2.2 the exclusion process is Bernoulli thus very weak Bernoulli. So we can choose $N_0 > J_0$ large enough so that line 16 is satisfied for all $N \geq N_0/p'$, with ϵ replaced with $\epsilon^2/(2n+1)$ and r replaced with ϵ.

Fix $N \geq N_0/\rho'$. Now we define S to be all (ω, y) which code well for M. There exists a set of relative measure $1-\epsilon$ in $Q_{\omega,N}$ of points which code well for M, and $(\omega, y) \in G$ from the definition of very weak Bernoulli. Thus $m(S) > 1-\epsilon$. Fix $(\omega, y) \in S$. Since $(\omega, y) \in G$

$$\bar{d}^Q_{[0,\epsilon N/\rho'] \times [0,N/\rho']}(m, (m|_{Q_{\omega,N}})) < \epsilon^2/(2n+1).$$

Thus the number of $i \in (0, \epsilon\rho' N)$ so that

$$\bar{d}_{[i-n,i+n] \times [0,N/\rho']}(m, (m|_{Q_{\omega,N}})) < 3\epsilon^2/(2n+1).$$

is at least $\epsilon N/2\rho'$. By the previous line and condition 2 of the definition of codes well for M there is at least one i so that

$$\bar{d}^Q_{[i-n,i+n] \times [0,N/\rho']}(m, (m|_{Q_{\omega,N}})) < 3\epsilon^2/(2n+1) \tag{18}$$

and

$$m|_{Q_{\omega,n}}(\{(\omega_1, y_1) : \ \bar{f}_{[0,N]}(M(T^i(\omega_1, y_1)), \Phi^*(T^i(\omega_1, y_1))) < \epsilon\}) > .25. \tag{19}$$

Line 18 and Lemma 5.8 imply that there exists a coupling $\tilde{c}$ of $(m|_{Q_{\omega,N}})$ and m such that for the given i

$$\tilde{c}(\{(\omega_1, y_1)(\omega_2, y_2) : \ \bar{f}_{[0,N]}(M(T^i(\omega_1, y_1)), M(T^i(\omega_2, y_2))) < 2\epsilon\}) > 1-2\epsilon. \tag{20}$$

Combining lines 19 and 20 we get that

$$\tilde{c}(\{(\omega_1, y_1)(\omega_2, y_2) : \ \bar{f}_{[0,N]}(\Phi^*(T^i(\omega_1, y_1)), M(T^i(\omega_2, y_2))) < 3\epsilon\}).22. \tag{21}$$

By condition 2 of Lemma 5.8 we have that

$$m(\{(\omega_2, y_2) : \bar{f}_{[0,N]}(M(T^i(\omega_2, y_2)), \Phi^*(T^i(\omega_2, y_2))) < \epsilon\}) > 1-\epsilon.$$

This also implies that

$$\tilde{c}(\{(\omega_1, y_1)(\omega_2, y_2) : \ \bar{f}_{[0,N]}(\Phi^*(T^i(\omega_1, y_1)), \Phi^*(T^i(\omega_2, y_2))) < 4\epsilon\}) > .2.$$

Thus $\nu_{T^i(\omega),N}$ can be coupled with ν' so that 1/5 of the mass is on points within 4ϵ in $\bar{f}_{[0,N]}$. As $i < \epsilon N$ we have that there exists a coupling c of ν' and $\nu_{\omega,n}$ with

$$c(\{(y_1, y_2) : \ \bar{f}_{[0,N]}(y_1, y_2) < 5\epsilon\}) > .2. \tag{22}$$

□

Proof of Lemma 5.1: Suppose Lemma 5.1 is not true. Then the conclusions of Lemmas 5.9 and 5.11 are true. Thus we can choose ϵ and N such that lines 15 and 17 are satisfied as well as the inequality

$$\nu' \times \nu'(\{(x, x') : \ \bar{f}_{[0,N]}(x, x') > 20\epsilon\}) > 1 - \epsilon. \tag{23}$$

There exists $y, y', z, z' \in \{\text{red, blue}\}^N$ such that

$$(y, y') \in A = \{(y_1, y_2) : \ \bar{f}_{[0,N]}(y_1, y_2) < 5\epsilon\},$$

$$(y, z), (y', z') \in \{(x, x') : \ \bar{f}_{[0,N]}(x, x') > 20\epsilon\}$$

and

$$(z, z') \in \{\omega : \ \nu_{\omega,N} \text{ is } \epsilon \text{ contained in an } \epsilon \ \bar{f}_{[0,N]} \text{ neighborhood}\}.$$

As $(y, y') \in A$

$$\bar{f}_{[0,N]}(y, y') > 20\epsilon$$

but

$$\bar{f}_{[0,N]}(y, z) < 5\epsilon$$

and

$$\bar{f}_{[0,N]}(y', z') < 5\epsilon$$

and

$$\bar{f}_{[0,N]}(z, z') < 5\epsilon.$$

This is a contradiciton. □

It follows easily from Lemma 5.1 that the average speed is an isomorphism invariant.

Corollary 5.1 *If the colored exclusion process generated by (α, ρ, c) is isomorphic to the colored exclusion process generated by (α', ρ', c') then $s(\alpha, \rho, c) = s(\alpha', \rho', c')$.*

Proof: If $s(\alpha, \rho, c) \neq s(\alpha', \rho', c')$ then there can be no isomorphism Φ that satisfies the conclusion of Lemma 5.1. Thus the two processes are not isomorphic. □

6 $r(\alpha, \rho, c)$ is an isomorphism invariant

We will use Lemma 5.1 to show that $r(\alpha, \rho, c)$, the relative one dimensional entropy of the coloring, is an isomorphism invariant. Here is a sketch of the proof. First we find a finite code Φ_n which is a very good approximation of Φ. We want N to be much larger than n, the size of the finite code. Since Φ_n approximates Φ well, $\Phi^*_n(\omega, y)$ determines $\Phi^*(\omega, y)$ up to a small error in $\bar{f}$.

We will show that $\Phi^*(\omega, y)$ depends mostly on y. The values $\Phi^{*,t}_n(\omega, y)_i$, $0 \le i \le N$, approximate the color of some particles in $\Phi^*(\omega, y)$ up to a small error in $\bar{f}$. Because of Lemma 5.1, we can choose N large enough so that a fixed fraction of the time $|d^t_n| < \epsilon N$. This ensures that for a fixed fraction of the time $\Phi^{*,t}_n(\omega, y)_i$, $0 \le i \le N$, is approximating $\Phi^*(\omega, y)_i$, $0 \le i \le N$ (in $\bar{f}$). The fact that $\Phi^{*,t}_n(\omega, y)$ depends mostly on y will imply that the relative entropy is an isomorphism invariant.

Given ϵ_0, ϵ and N we say a point (ω, y) **codes well for** Φ^*_n if there is a set $S = S(\omega, y) \subset \mathbb{Z}$ of upper density at least $9\epsilon_0$ such that

1. for all $t \in S$, $|d^t_n| < \epsilon\rho' N$
2. for all $t \in S$
$$\bar{f}_{[0,\rho' N]}(\Phi^*(\omega, y), \bar{\Phi}^t_n(\omega, y)) < 2\epsilon, \tag{24}$$
3. $\bar{f}_{[0,\rho' N]}(\Phi^*(\omega, y), \Phi^*_n(\omega, y)) < \epsilon$, and
4. $\omega^{\rho N} > (F(\Phi(\omega, y)))^{(1-\epsilon)\rho' N} + 2n$.

Note that conditions 1, 2 and 3 imply that for all $t \in S$

$$\bar{f}_{[0,\rho' N]}(\Phi^*_n(\omega, y), \Phi^{*,t}(\omega, y)) < 4\epsilon.$$

Lemma 6.1 *There exists $\epsilon_0 > 0$ such that for any $\epsilon > 0$ there exists N such that the measure of points that do code well for Φ^*_n is at least $5\epsilon_0$.*

Proof: By Lemma 5.1 we can choose ϵ_0 so that for all N sufficiently large and for all T

$$m(\{(\omega, y) : \ \#\{t : \ t \in [0, T] \text{ where } |d^t_n(\omega, y)| < \epsilon\rho' N\} \le 10\epsilon_0/(T+1)\} \le 1 - 10\epsilon_0. \tag{25}$$

Without loss of generality $\epsilon < \epsilon_0$. By Lemma 5.2 we can choose Φ_n and J_0 so the set of (ω, y) such that for all $J > J_0$

$$\bar{f}_{[0,J]}(\Phi^*_n(\omega, y), \Phi^*(\omega, y)) < \epsilon \tag{26}$$

has measure $> 1-\epsilon$. We also want $J_0 < \epsilon\rho' N$, $n < \epsilon\rho' N$ and for Lemma 5.4 to be satisfied.

Condition one is satisfied for a set of measure at least $10\epsilon_0$ by line 25. By Lemma 5.4 the measure of points that satisfy the second condition is at least $1-\epsilon$. The third condition is satisfied by a set of measure at least $1-\epsilon$ by line 26. The strong law of large numbers for a Markov chain implies that for large N the last condition is satisfied on a set of at least measure $1-\epsilon$. Thus if N is sufficiently large the measure of points that do code well for Φ^*_n is at least $7\epsilon_0$. □

Given ϵ, ϵ_0 and N define $H : \{\text{red,blue}\}^{N\rho} \to \{\text{red,blue}\}^{N\rho'}$ as follows. For each $a_i \in \{\text{red, blue}\}^{N\rho}$ choose a point (ω, y) that codes well for Φ^*_n so that $a_i = y_i, 0 \leq i \leq N\rho$, and (ω, y) satisfies the ergodic theorem for all cylinder sets in the free range process, if possible. Then define $H(y)_j = \Phi^*_n(\omega, y)_j$.

Lemma 6.2 *For all $\epsilon > 0$ there exists an N large enough and $G \subset \Omega \times Y$, $m(G) > 2\epsilon_0$, such that if $(\omega', y') \in G$ then*

$$\bar{f}_{[0,\rho' N]}(\Phi^*(\omega', y'), H(y')) < 10\epsilon.$$

Proof: Given (ω', y') let (ω, y) be the point used in the definition of $H(y'_0, ..., y'_{\rho N})$, if it exists. We say $(\omega', y') \in G$ if

1. satisfies conditions 3 and 4 in the definition of codes well for Φ^*_n and
2. there exists a $t \in S(\omega, y)$ so that

$$\omega'_{i,j} = \omega_{\omega^{0,t}+i,t+j} \tag{27}$$

   for all $|j| \leq n$ and $0 \leq i \leq \omega^{\rho N}$.

Given such an (ω', y') we get $t \in S(\omega, y)$. Since (ω, y) codes well for Φ^*_n and $t \in S$

$$\bar{f}_{[0,\rho' N]}(\Phi^{*,t}_n(\omega, y), \Phi^*_n(\omega, y)) < 4\epsilon. \tag{28}$$

Since (ω', y') satisfies line 27 and condition 4 of the definition of codes well for Φ^*_n, and (ω, y) codes well for Φ^*_n then

$$\Phi^*_n(\omega', y')_i = \Phi^{*,t}_n(\omega, y)_i$$

for all i such that $2n \leq i \leq (1-\epsilon)\rho' N$. Combined with line 28 and the fact that $H(y') = \Phi^*_n(\omega, y)$ this implies

$$\bar{f}_{[0,\rho' N]}(\Phi^*_n(\omega', y'), H(y')) < 6\epsilon.$$

Because (ω', y') satisfies condition 3 of coding well for Φ^*_n we have that

$$\bar{f}_{[0,\rho' N]}(\Phi^*(\omega', y'), \Phi^*_n(\omega', y')) < \epsilon.$$

Putting this together gives

$$\bar{f}_{[0,\rho' N]}(\Phi^*(\omega', y'), H(y')) < 7\epsilon.$$

□

Lemma 6.3 *If the colored exclusion process generated by* (α, ρ, c) *is isomorphic to the one defined by* (α', ρ', c') *then* $\rho h(c) = \rho' h(c')$.

Proof: Suppose $\rho' h(c') - \rho h(c) = \delta > 0$. Get ϵ_0 from Lemma 6.1. Then choose N and ϵ so that any set A such that $\nu'(A) > \epsilon_0/2$ cannot be covered by

$$2^{(h(c')\rho' - \delta/2)N} = 2^{(h(c)\rho + \delta/2)N}$$

neighborhoods of $\bar{f}_{[0,\rho' N]}$ diameter 10ϵ. We also want N and ϵ so that there exists a set B such that $\nu(B) > 1 - \epsilon_0/2$ which can be covered by $2^{(h(c)\rho+\delta/2)N}$ cylinder sets of length ρN. Finally we get G from Lemma 6.2.

This is possible by the Shannon-McMillan-Breiman theorem and the fact that the ν' measure of an 10ϵ $\bar{f}_{[0,\rho' N]}$ neighborhood of alomst any y' can be made less than

$$2^{-(h(c')\rho' - \delta/2)N} = 2^{-(h(c)\rho + \delta/2)N}$$

if ϵ is small enough and N is large enough. Thus there exists a subset $S \subset B \cap G$ with $\nu(S) > \epsilon_0/2$ which can be covered by $2^{(h(c)\rho+\delta/2)N}$ cylinder sets and to which Lemma 6.2 applies. By applying Lemma 6.2 there exists a set $H(S)$ with $\nu'(H(s)) > \epsilon_0/2$ and $H(S)$ is covered by $2^{(h(c)\rho+\delta/2)N}$ 10ϵ $\bar{f}_{[0,\rho' N]}$ neighborhoods. This is a contradiction. The other inequality comes from the same analysis applied to Φ^{-1}. □

Proof of Theorem 3.1: The entropy of the colored exclusion process is an isomorphism invariant. Corollary 5.1 implies that the speed is also an isomorphism invariant. Lemma 6.3 says the relative entropy of the colored exclusion process is an isomorphism invariant. □

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