

The behavior of Bernoulli shifts relative to their factors

Christopher Hoffman

November 4, 1998

Abstract

We present numerous examples of ways that a Bernoulli shift can behave relative to a family of factors. This shows the similarities between the properties which collections of ergodic transformations can have and the behavior of a Bernoulli shift relative to a collection of its factors. For example, we construct a family of factors of a Bernoulli shift which have the same entropy, and any extension of one of these factors has more entropy, yet no two of these factors sit the same. This is the relative analog of Ornstein and Shields uncountable collection of nonisomorphic K transformations of the same entropy. We are able to construct relative analogs of almost all the zero entropy counterexamples that Rudolph constructed in [20], as well as the K counterexamples constructed in [6]. This paper provides a solution to a problem posed by Ornstein in [14].

1 Introduction

In this paper we are trying to further the analogy between ergodic measure preserving transformations and the behavior of a Bernoulli shift relative to its factors. One of the most important results of the study of the behavior of a transformation relative to its factors is Thouvenot's relative isomorphism theorem. This generalization of Ornstein's isomorphism theory gave a criteria for answering the following question. Given a transformation and a factor, when can the transformation be written as the direct product of that factor and a Bernoulli shift? As the criteria is closely related to Ornstein's very weak Bernoulli condition, we say that if a transformation and a factor satisfy Thouvenot's condition, then the transformation is relatively very weak Bernoulli with respect to the factor.

To show the contrast between Bernoulli shifts and other transformations, there are a large number of examples of families of K or mixing transformations which have properties that the isomorphism theorem proves that families of Bernoulli shift can not have. For example, Ornstein and Shields created an uncountable family of nonisomorphic K transformations of the same entropy [12]. A general method for constructing families of zero entropy mixing and loosely Bernoulli transformations with properties that a family of Bernoulli shifts cannot have is given in [5], and is a variation of a method created by Rudolph in [20]. A general method for constructing families of K and loosely Bernoulli transformations is given in [6]. In this paper we will show that in a Bernoulli shift there exists a similar difference in the way that a Bernoulli shift behaves relative to relatively very weak Bernoulli factors and not relatively very weak Bernoulli factors.

There are natural properties of factors that are analogous to properties of transformations. For example, the relative analog of two transformations being isomorphic is if two factors of the same transformation sit the same. We say that two factors in a transformation **sit the same** if there exists an automorphism of the transformation which takes one factor to the other. The goal of this paper is, for each counterexample alluded to in the previous paragraph, to construct the analogous relative counterexample. That is a family of factors of a Bernoulli shift that have the analogous properties of the

example. An example of this is an uncountable family of factors of the same entropy in a Bernoulli shift that do not sit the same, but any extension of one of the factors has more entropy than the factor. This is the relative analog of Ornstein and Shields uncountable family of nonisomorphic K transformations that have the same entropy.

Besides Thouvenot's relative isomorphism theorem, the analogy between ergodic transformations and the behavior of a Bernoulli shift relative to its factors has been studied by Rudolph. In a series of papers he expanded this analogy by classifying all of the relatively compact factors of a Bernoulli shift [17]. In particular, he showed that all two point factors of a Bernoulli shift sit the same [19]. This corresponds to the fact that there is only one ergodic action on a two point space. In contrast to this, he showed that a mixing transformation can have an uncountable collection of two point factors, no two of which sit the same [20]. This example shows that the analogy between ergodic transformations and the behavior of a Bernoulli shift relative to its factors does not work if we replaced the Bernoulli shift by a transformation which is not isomorphic to a Bernoulli shift.

The first relative counterexample was constructed by Ornstein in [14]. This was a factor of a Bernoulli shift that is relatively K , but the whole transformation can not be expressed as the direct product of the factor and another factor. This is the relative analog of a K but not Bernoulli transformation. Much of this paper is based on that example. Not many other relative counterexamples have been constructed. Swanson proved in [21] that a Bernoulli shift has uncountably many partitions that generate relatively K factors but the partitions do not sit the same. (See section 6 for the definition of a partition sitting the same.) We will show how to strengthen this result and prove that there exists an uncountable collection of relatively K factors that do not sit the same. For almost every counterexample created among K transformations in [6], and every zero entropy loosely Bernoulli counterexample created in [5], we will construct an analogous relative counterexample.

The analogy runs deeper than simply that corresponding relative counterexamples exist. We will use the zero entropy transformations in [5], and the K transformations in [6], to construct the analogous relative counterexamples. To do this we will take skew products of a Bernoulli shift with these transformations. We will show the skew products are Bernoulli shifts of the same entropy, and therefore they are all isomorphic. To construct the family of factors we will look at the images of the base factors under the isomorphisms. With each skew product there will be an associated permutation. We will reduce the question of whether two base factors sit the same to whether the two associated permutations are conjugate. This is done in two steps. Using coding arguments we will be able to show that if there is an isomorphism between two of the skew products which is the identity on base factor, then the two corresponding permutations are conjugate. In section 9, we will use this result to show that any automorphism that preserves the base factor generates an automorphism of a symmetric group. This, and the fact that almost every automorphism of a symmetric group is an inner automorphism, will tell us that the two base factors sit the same if and only if the two associated permutations are conjugate.

2 Construction of the Zero Entropy and K Transformations.

The zero entropy and K transformations will be constructed by similar cutting and stacking arguments. We start with the zero entropy transformations. Let Q be the partition $(0, s)$. The zero block, $B(0)$, is the single symbol 0. The n block, $B(n)$, will be a string of $h(n)$ symbols of Q . To construct $B(n)$ inductively from $B(n-1)$ we need to choose $N(n)$, the number of $n-1$ blocks in the n block. We also need c_n , the number of cyclic regions at the beginning of an n block, and x_n , where $x_n N(n) \in \mathbf{Z}$ is the number of $n-1$ blocks in each cyclic region. We must have $c_n x_n \leq 1$. To specify the number of spacer s

symbols between the $n - 1$ blocks we need two sequences of integers. The first, $p_{1,n}, p_{2,n}, \dots, p_{c_n,n}$, tells how many spacer symbols to put after an $n - 1$ block in each of the cyclic regions. The second is the pseudorandom sequence, $a_{1,n}, a_{2,n}, \dots, a_{(1-c_n x_n)N(n),n}$. We also need $S(n)$ so that all of the $a_{i,n}$ and $p_{i,n}$ are between 1 and $S(n)$. If these are all fixed then we construct the n block in the following way.

Let the j th cyclic region, $\overline{B_{j,n}}$, be defined as follows.

$$\overline{B_{j,n}} = (n - 1 \text{ block}) \overbrace{sssss}^{p_{j,n}} (n - 1 \text{ block}) \overbrace{sssss}^{p_{j,n}} \dots (n - 1 \text{ block}) \overbrace{sssss}^{p_{j,n}}.$$

Each cyclic region has $x_n N(n)$ $n - 1$ blocks. Now we put these together in order to form the cyclic portion of the n block. To form the mixing portion of the n block we define

$$B_{j,n}^* = \overbrace{sssss}^{a_{i,n}} (n - 1 \text{ block}) \overbrace{sssss}^{S(n)-a_{i,n}}.$$

The mixing portion of an n block is created by putting those together in order. Then the n block is the cyclic portion followed by the mixing portion,

$$B(n) = \overline{B_{1,n}} \dots \overline{B_{c_n,n}} B_{1,n}^* B_{2,n}^* \dots B_{(1-c_n x_n N(n)),n}^*.$$

In order for names to have long enough cyclic regions so that the skew products will be very weakly Bernoulli we choose $x_n \in \mathbf{Q}$ so that

$$x_n \rightarrow 0 \text{ and } \sum_{n=1}^{\infty} (x_n)^p = \infty \text{ for all } p.$$

For technical reasons we choose $x_n = 0$ for all $n = 0, 2 \pmod 3$. Next we select $c_n \in \mathbf{Z}$ increasing to infinity but doing so slowly enough that $c_n x_n \rightarrow 0$.

We will use a sequence ϵ_n such that $\sum_N^\infty \epsilon_n < \epsilon_{N-1}$. Finally we choose $S(n+1), N(n), a_{i,j}$, and $p_{i,n}$ inductively. We choose $S(n+1)$, so large that $\frac{h(n-1)}{S(n+1)} < \epsilon_n$.

Now we pick $N(n)$. Our first restriction on $N(n)$ is in order to make sure the measure space, Ω is finite. We require that the number of spacers used in building the n block are a small fraction of the length of an n block. To do this, we need

$$\frac{S(n+1)}{h(n)} < \frac{S(n+1)}{N(n)h(n-1)} < \epsilon_n.$$

For any choice of the parameters listed above we will get a transformation. We want to make our choices so that the resulting transformation has a special mixing property for each n . First we need a few definitions.

We say that a point ω **is in the n block** if for some $i, i \leq 0 \leq i + h(n) - 1$ the string of symbols $Q(T^i(\omega)), \dots, Q(T^{i+h(n)-1}(\omega))$ form the n block. We call that interval of integers $(i, i + h(n) - 1)$ **the n block around 0 for ω** . Let the interval $(i, i + h(n) - 1)$ be the n block around 0 for x and the interval $(j, j + h(n) - 1)$ be the n block around 0 for y . These two n blocks are said to **line up** if $|j - i| < (h(n - 1) + S(n))/l(t)$. Let l_1, \dots, l_k be nonzero integers. We now generalize this terminology to the case where we have a transformation $U = T^{l_1} \times \dots \times T^{l_k}$ acting on Ω^k . We say that the k fold n overlap around 0 for $\omega_1, \dots, \omega_k$, is the largest interval of integers $(-j, j')$ such that for all i , and $t \in (-j, j')$, $T^{tl_i}(\omega_i)$ is in the n block around 0. Similarly the mixing k fold n overlap around 0, $P_{0,n}(\omega_1, \dots, \omega_k)$, is defined to be the largest interval of integers $(-j, j')$ such that for all i and $t \in (-j, j')$ $T^{tl_i}(\omega_i)$ is in the mixing portion of the n block.

We want to choose $N(n)$ large enough so that there exists a pseudorandom sequence, $a_{i,n}$, such that the n block has the following mixing property. For any k fold mixing overlap, (i, j) , of n blocks, with $0 \leq k, |l_i| \leq n$, and for $1 \leq z_1, z_2, \dots, z_k \leq h(n - 2)$, one of three things must happen. Either

1. the mixing overlap is extremely short ($< \epsilon_n h(n)$),
2. there are points ω_t and $\omega_{t'}$, such that $t = t'$ and the n blocks for ω_t and $\omega_{t'}$ line up, or
3. $|\frac{1}{j-i-1}(\# \text{ of } t|T^{l_1 t}(\omega_1) \text{ is in the } z_1 \text{th position of an } n-2 \text{ block,}$
 $T^{l_2 t}(\omega_2) \text{ is in the } z_2 \text{th position of an } n-2 \text{ block,} \dots$
 $T^{l_k t}(\omega_k) \text{ is in the } z_k \text{th position of an } n-2 \text{ block})$
 $-1/h(n-2)^n| < \epsilon_n$

Lemma 2.1 (Rudolph) *There exists an N , such that for all $N(n) > N$, there exists a sequence $a_{i,n}$, $1 \leq i \leq N(n)(1 - x_n c_n)$, which satisfy the above condition on mixing n overlaps.*

Proof: This is an application of the exponential rate of convergence for the weak law of large numbers and is proved by Rudolph [20]. \square

Corollary 2.1 *There exists a sequence $a_{i,n}$ and a N such that for any $N(n) > N$, $a_{1,n}, \dots, a_{1-x_n c_n N(n),n}$ gives mixing n overlaps the property described above.*

Proof: This is true because as N increases the percentage of sequences of length N which don't have the desired property is decreasing exponentially. This is also proved in [20]. \square

We also need to have $x_n N(n) \in Z$, and if $x_n > 0$

$$\frac{(2h(n-1))^{c_n}}{x_n N(n) h(n-1)} < \epsilon_n.$$

Choose $N(n)$ so that all of the above conditions are satisfied. Then select a sequence $a_{i,n}$, $1 \leq i \leq N(n)(1 - x_n c_n)$, that satisfies our condition on mixing n overlaps. Now choose $p_{1,n+1}, \dots, p_{c_{n+1},n+1}$ so that $h(n) + p_{1,n+1}, \dots, h(n) + p_{c_{n+1},n+1}$ are relatively prime and $\frac{p_{j,n+1}}{h(n)} < \epsilon_n$. This can be done by a lemma in [4]. Proceeding in this manner we select $N(n)$, $S(n)$, $a_{i,n}$, and $p_{i,n}$. After all of these have been chosen, this defines our measure space, Ω . Our transformation, T , is translation.

To construct our basic K automorphism we will use a similar cutting and stacking approach. The block structure is almost identical to the zero entropy block structure. Let Q be the partition $(e, f, s, 0)$. The only difference is the random number of the symbols f and e at the beginning and end of the blocks. The 0 block consists of one 0. An n block begins with a string of f . This is followed by a series of $N(n)$ $n-1$ blocks, where each pair of consecutive $n-1$ blocks may be separated by a number of s . It ends with a string of e .

$$\underbrace{\overbrace{fff}^F (n-1) \dots (n-1) (n-1) s \dots (n-1) s \dots (n-1) s s s \dots (n-1) \overbrace{sss}^{p_{c_n,n}} \overbrace{ssss}^{a_{i,n}} (n-1) \overbrace{sssssss}^{S(n)-a_{i,n}} \dots s s s s s s s s (n-1) s s s \overbrace{eeeeeeee}^{2h(n-1)+n-F}}}_{\text{cyclic region} \quad \text{mixing region}}$$

The number of f at the beginning of an n block, F , will be chosen with uniform distribution from $h(n-1) + 1$ to $h(n-1) + n - 1$. The number of e will be $2h(n-1) + n - F$. Corresponding to every choice of F and every sequence of $N(n)$ $(n-1)$ blocks is one n block. Thus if there are $A(n-1)$ different $n-1$ blocks, there are $A(n) = (A(n-1))^{N(n)} (n-1)$ n blocks all of length $h(n)$. We do the construction so that all of our n blocks have equal measure.

Lemma 2.2 *There exists an N_0 such that for any $N(n) > N_0$ there is an appropriate "pseudorandom" sequence, $a_{c_n x_n + 1}, \dots, a_{N(n)}$. That is, for any points $(\omega_1, \omega_2, \dots, \omega_k)$ any mixing overlap, (i, j) of n blocks, with $0 \leq k \leq n$, and consecutive strings of integers $Z_1, Z_2, \dots, Z_n \subset (1, \dots, h(m))$ $m \leq n-2$ one of three things must happen. Either*

- 1) the mixing overlap is extremely short ($j - i < \epsilon_n h(n)$),
- 2) there are two points ω_t and $\omega_{t'}$ such that the n blocks for ω_t and $\omega_{t'}$ line up, or
- 3) $|\frac{1}{j-i-1}(\# \text{ of } t|K^t(\omega_1) \text{ is in the } i_1 \text{th position of an } m \text{ block for some } i_1 \in Z_1,$

$$\begin{aligned}
& K^t(\omega_2) \text{ is in the } i_2 \text{th position of an } m \text{ block for some } i_2 \in Z_2, \dots, \\
& K^t(\omega_k) \text{ is in the } i_k \text{th position of an } m \text{ block for some } i_k \in Z_k \\
& - \prod (|Z_i|/h(n-2)) \\
& < (\prod (|Z_i| + n(n+1)/2) - \prod |Z_i|)/h(n-2)^k + \epsilon_{n-2}.
\end{aligned}$$

Proof: This follows from lemma 2.1. The only thing we have to account for is the amount the different choices of f can affect the mixing properties of the pseudorandom sequence. \square

There is one other property of the K transformation that we will need.

Definition 2.1 Define $l(n)$ to be the infimum over any two m blocks, $m > n$, that overlap in a length at least $\beta_m h(m)$ but do not line up, of the fraction of the n blocks in the overlap of the two m blocks that do line up.

Lemma 2.3 $l(n) \rightarrow 0$ as $n \rightarrow \infty$.

Proof: This is contained in [6]. \square

3 Construction of the Base Transformation

The construction of the base is done by cutting and stacking and is a variant of the base transformation in [14]. Let P be the partition $(0, 1, e, f, s)$. There is only one zero block and it consists of the single symbol 0. If there are $A(n)$ different n blocks each with length $H(n)$ and all of the same measure, then there will be $A(n+1) = A(n)^{M(n+1)} F(n+1) G(n+1)$ different $n+1$ blocks, all with the same measure. $F(n+1)$ is the total number of f and e at the beginning and end of the $n+1$ block. $G(n+1)$ is the number of 0s and the number of 1s at the beginning (and the number at the end) of an $n+1$ block. $M(n+1)$ is the number of n blocks in an $n+1$ block. Label the n blocks from 1 to $A(n)$. Corresponding to each choice of the $A(n+1)$ integer choices of $f, g, b_1, \dots, b_{M(n)}$ such that $1 \leq f \leq F(n)$, $1 \leq g \leq G(n)$, and $1 \leq b_1, \dots, b_{M(n)} \leq A(n)$ is one $n+1$ block which has the following form.

$$\underbrace{00000}_g \underbrace{11111111}_{G(n+1)-g} \underbrace{ffff}_{f} (\text{n block } b_1) \dots (\text{n block } b_{M(n)}) \underbrace{eeeeeeee}_{F(n+1)-f} \underbrace{11111}_g \underbrace{00000000}_{G(n+1)-g}$$

Now we need to specify $M(n), F(n)$, and $G(n)$ for all n in order to complete the description of the transformation. We want $F(n), G(n) \gg H(n-1)$, $x_n h(n+1) \gg H(n) > n \epsilon_{n+1} h(n+1)$, and for all k , $G(n) \gg h(n)^k$. Where $h(n+1)$ is the length of the zero entropy or the K $n+1$ block. We use the notation $a(n) \ll b(n)$ to mean that $\sum a(n)/b(n) < \infty$. For now take any $F(n), G(n)$, and $N(n)$ so that these conditions are satisfied.

4 Construction of the Skew Products

Now that we have defined our base transformation B we are ready to define a family of skew products. Let π be a permutation on V , a finite or countable set, that has only cycles of finite length. Then we will define $B \times_{\pi} S : \Omega' \times (\Omega)^V \rightarrow \Omega' \times (\Omega)^V$ as follows

$$B \times_{\pi} S(b, y_1, y_2, \dots) = \begin{cases} (B(b), S(y_{\pi(1)}), S(y_{\pi(2)}), \dots) & \text{if } P(x) = 0 \\ (B(b), y_{\pi(1)}, y_{\pi(2)}, \dots) & \text{if } P(x) \neq 0 \end{cases}$$

where S is either our zero entropy transformation or our K transformation.

Definition 4.1 Our base factor in $B \times_{\pi} S$ associates any two points of the form (x, y_1, y_2, \dots) and (x, z_1, z_2, \dots) .

If we use our zero entropy transformation to form the $B \times_\pi S$, then it is easy to see that the whole transformation has relative zero entropy with respect to the base factor. If we use our K transformation to form the $B \times_\pi S$, then we have the following lemma.

Lemma 4.1 *The base factors are relatively K .*

Proof: First assume V is finite. Pick any k , any $\epsilon > 0$, ϵ almost any base name b , and any K name past, P , specifying the Q^V name up to time 0. Then we need to show that there exists a T such that the conditional distribution during times T to $T+k$ given b and P is within 10ϵ of the conditional distribution during times T to $T+k$ given b . This condition is equivalent to the base factor being relatively K by a relative version of the Pinsker-Sinai-Rokhlin theorem [15].

First pick an n such that the measure of points in n blocks for the K names are greater than $1 - \epsilon^2/(k+1)|V|$. This tells us that the conditional distribution of $|V|$ strings of length $k+1$ all in n blocks is within ϵ^2 of the unconditional distribution. Next choose an m such that $\epsilon(m-1) > h(n)$ and $M = h(m)$. Also choose a T such that ϵ almost every base name has had at least M zeros between time 0 and time T . Now P and b may combine to tell us whether $(T, T+k)$ is in an m block, and where it is inside of an m block, but it tells us nothing about which m block it is in. Restrict to those bases and pasts that tell us that $(T, T+k)$ is with probability at least $1 - \epsilon$ inside of an n block and that there have been at least M zeros in the base name from time 0 to time T . This eliminates a set of bases and pasts of measure less than 2ϵ . Since we don't know the number of f at the beginning of the m blocks that $(T, T+k)$ is in and $\epsilon(m-1) > h(n)$ we could be anywhere within an n block with ϵ almost equal probability. Since we could be in any n block with equal probability, independent of where we are in that n block the conditional distribution given P and b is within ϵ of the conditional distribution given b , and that we are in an n block which is within ϵ of the conditional distribution given b from times T to $T+k$. Since finite $|V|$ give us a generating sequence of partitions we conclude that the base factor is relatively K for infinite $|V|$ as well. \square

5 The Skew Products are Bernoulli

We want to show that all of our skew products, $B \times_\pi S$, are Bernoulli. First we must show that our base transformation is Bernoulli. This is done by a nesting procedure. Given two n blocks that don't line up we will use the varying number of f at the beginning of an n blocks to line up a fixed fraction of the $n-1$ block structures. On the $n-1$ blocks that we were not able to line up, we repeat this procedure to line up the same fixed fraction of $n-2$ blocks. We continue until almost all of the k block structures line up for some k . We then fill those spots with the same k blocks.

Lemma 5.1 *The base transformation (B, Ω') is VWB.*

Proof: Given $\epsilon > 0$ we want to choose n and k large enough so that $(2/3)^k < \epsilon^2$ and points in n blocks make up all but ϵ of Ω' . Choose N so that $h(n+k)/N < \epsilon$ and all but ϵ of the points in Ω' have $(0, N)$ in an $n+k+1$ block. Take any two pasts that force $(0, N)$ to be in an $n+k+1$ block. These pasts tell us the position of the $n+k$ blocks from the time 0 to N and something about what is inside the first partial $n+k$ block, but nothing about how the subsequent blocks are filled in. Compare the structure of the $n+k$ blocks from the two pasts and pair the $n+k$ blocks up so that the overlap of paired blocks is at least $(1/2)h(n+k)$. Choose the f at the beginning of the paired $n+k$ blocks so that on the overlap of these paired blocks the $n+k-1$ blocks line up perfectly. For any number of f between $h(n+k)$ and $F(n+k) - h(n+k-1)$ such a match is possible. Averaging over all blocks we have matched at least $1/3$ of the $n+k-1$ block structures perfectly. Fill in these matched block structures with the same $n+k-1$ block in each name.

On the $n+k-1$ blocks that are not matched perfectly we are going to repeat this procedure. Choose the number of f at the beginning of the $n+k-1$ blocks so that at least $1/3$ of the $n+k-2$ blocks line up. Fill those blocks in the same on both names. Proceeding in this manner we can match at least $1/3$ of the blocks that were unmatched after the previous step. After k iterations of this procedure we will have matched a fraction greater than $1 - (2/3)^k > 1 - \epsilon^2$ of the n block structures perfectly. Thus on most names most blocks have been matched. Since we filled in those structures with the same n blocks and n blocks make up at least $1 - \epsilon$ of the string from 0 to N with a 2ϵ good \bar{d} matching. \square

We will now do the matching which shows that the skew products, $B \times_\pi S$, are very weak Bernoulli. The lemma we prove below applies to the skew products constructed with either the zero entropy transformation or the K transformation. Remember that $\overline{B_{i,3n+1}}$ is the i th ‘‘cyclic’’ section of a $3n+1$ block. A $\overline{B_{i,3n+1}}$ block for x is a maximal consecutive string of integers such that for all i in the block $T^i(x) \in \overline{B_{i,3n+1}}$ for the zero entropy case (or $K^i(x) \in \overline{B_{i,3n+1}}$ for the K transformation).

Define $M_{3n+1,k}$ to be all (x_1, \dots, x_k) such that

$$\begin{aligned} x_1 &\in \overline{B_{1,3n+1}} \text{ and } (-H(3n), H(3n)) \text{ is a } \overline{B_{1,3n+1}} \text{ block} \\ x_2 &\in \overline{B_{2,3n+1}} \text{ and } (-H(3n), H(3n)) \text{ is a } \overline{B_{2,3n+1}} \text{ block. . .} \\ x_k &\in \overline{B_{k,3n+1}} \text{ and } (-H(3n), H(3n)) \text{ is a } \overline{B_{k,3n+1}} \text{ block.} \end{aligned}$$

Define $B_{3n,k}$ to be all points (y_1, \dots, y_k) in Ω^k such that for $i \neq j$ the $3n$ blocks around 0 for y_i and y_j do not line up to within $h(3n-1)$ and $(0, H(3n-1))$ is in an n block for all y_i . We define δ_{3n} as follows.

$$\mu(B_{3n,k}) > \frac{(1 - kH(3n-1))}{h(3n)} - \frac{k^2}{2N(3n)} = 1 - \delta_{3n}$$

Let $s_2 = 1$ and

$$s_{3n-1} = \left(\frac{x_{3n-2}}{2}\right)^{2k} \epsilon_{3n-3} + \left(1 - \left(\frac{x_{3n-2}}{2}\right)^{2k}\right) s_{3n-4} + 4\delta_{3n-3}.$$

We have that $s_{3n-1} \rightarrow 0$ because $\sum \delta_n < \infty$, $\sum x_n^{2k} = \infty$, and $\epsilon_n \rightarrow 0$.

Let p and q be atoms of $\Omega' \times \Omega^k$ such that they tell us about what happened in the past and present and what happens in the future after time $H(3n-1)$, but not any more about what happened between times 1 and $H(3n-1)$. We also want that for any point $(b, y_1, \dots, y_k) \in p$ or q that $(1, H(3n-1))$ is a base $3n-1$ block of b and that (y_1, \dots, y_k) is in $B_{3n,k}$.

Lemma 5.2 *For any p, q described above $\bar{d}_{[1, H(3n-1)]}(\Omega' \times \Omega^k|_p, \Omega' \times \Omega^k|_q) \leq s_{3n}$.*

Proof: We will prove the theorem by induction. The first step is trivial because $s_2 = 1$. Now assume that it is true for $3n-4$. We find $i, 0 \leq i \leq k^2 h(3n-1)$, such that for points $(b, y_1, \dots, y_k) \in p$ and $(b', y_1, \dots, z_k) \in q$ none of the $2k$ $3n$ blocks of a y_m and $K^i(z_p)$ line up. Then for at least $(x_{3n-2}/2)^{2k}$ of the j , $(S_{id})^j(p) \in M_{3n-2,k}$ and $(S_{id})^{j+i}(q) \in M_{3n-2,k}$. Call this set of j, J .

Fill in the 0 and 1 of corresponding $3n-1$ blocks so that paired q names have i more 1s at the beginning of the $3n-1$ block than the corresponding p names have. Since $G(n-1) \gg k^2 h(n-1)$ this works for most choices of the number of zero and ones. Fill in the 0 and 1 of corresponding $3n-2$ blocks the same. Fill in the f and e of corresponding $3n-1$ and $3n-2$ blocks the same. For the skew products with the K transformations fill in the f of corresponding $3n-1$ and $3n-2$ blocks the same.

Now for all base $3n-3$ blocks that are covered at one point by some $M_{3n,k}$ by both the p and q names we do the following matching. We use the differing number of zeros and ones at the start of the $3n-3$ block to line up the skewed names. Other than that fill in the rest of the bases and the skewed names identically. These blocks now match almost exactly. This percentage must be at least $(\frac{x_{3n-2}}{2})^{2k}$ by lemmas 2.1 or 2.2.

For the other base $3n-3$ blocks match the $f, e, 0$ and 1 the same. Most of the $3n-4$ blocks in these unmatched $3n-3$ blocks are covered entirely by top names that are in $B(3n-3)$ so we can use our

previous matching. On $1 - 4\delta_{3n-3}$ of the unmatched $3n - 4$ blocks are covered by $B_{3n-3,k}$ in both the p and q names. Thus we get

$$\begin{aligned} \bar{d}_{H(3n-1)}(\Omega' \times \Omega^k|_p, \Omega' \times \Omega^k|_q) &\leq \left(\frac{x_{3n-2}}{2}\right)^{2k} \epsilon_{3n-3} + \left(1 - \left(\frac{x_{3n-2}}{2}\right)^{2k}\right) s_{3n-4} + 4\delta_{3n-3} \\ &\leq s_{3n-1}. \end{aligned}$$

Lemma 5.3 *The skew products $B \times_\pi S$ are Bernoulli*

Proof: First we restrict to the same set of past base names as we did in the previous section. We also want top names that are in $B(3(n+k))$. Given two pasts we do the matching outlined in the previous section on the base names to get for some n almost all of the $3n - 1$ block structures line up, but, unlike in the previous section, we do not fill these in yet. For the skew products with the K transformations fill in the f of corresponding $3(n+k) - 1$ through $3n$ blocks the same.

Because the $3(n+k)$ blocks for each skewed name don't line up Lemmas 2.1 and 2.3 tell us only a small percentage of the $3n$ blocks line up. Thus most of the $3n$ block overlaps are in $B(3n)$. Most of the time from 0 to M is made up of base $3n - 1$ blocks which are covered by $3n$ overlaps in $B(3n)$ and that line up with a similar $3n - 1$ block in the other under the pairing. Now we can apply Lemma 5.2 to say that on most of these blocks we can get a good \bar{d} matching. Since these good base $3n - 1$ blocks make up most of the interval 0 to N we have a good \bar{d} matching.

Although the proof above was given for transformations where $\pi = id$, it works just as well for any π . Once we have matched the names without a permutation, applying the permutation will not screw up the \bar{d} matching.

This argument works for any skew product with a finite set of coordinates. To extend it to infinite sets of coordinates we take a refining sequence of partitions which generate skew products on finitely many coordinates and the span of these partitions generate the whole skew product. By the argument above, all of the finite partitions are VWB. The infinite skew product is VWB because the \bar{d} limit of VWB transformations is VWB. \square

6 Isomorphisms over the same Base

Two of the simplest skew products we created are $B \times_{id} S$ and $B \times_{(1,2)} S$ on $\Omega' \times \Omega^2$. By the results of the last section both of these two transformations are Bernoulli. Because they have the same entropy Ornstein's isomorphism theorem tells us that they are isomorphic [10]. There are two conditions stronger than an isomorphism that we may have.

Definition 6.1 *If there exist an isomorphism Φ that maps the factor generated by the base transformation in $B \times_\pi S$ to the factor generated by the base transformation in $B \times_{\pi'} S$, then we say the **base factor sits the same** in the two transformations. By this we mean there exists Φ such that if $\Phi(b, x_1, x_2) = (b', y_1, y_2)$ then for all points with the same base (b, z_1, z_2) $\Phi(b, z_1, z_2) = (b', w_1, w_2)$,*

Definition 6.2 *Suppose there exist an isomorphism Φ that is the identity on the factor generated by the base transformation, $B \times_\pi S(b, x_1, x_2) = (b, y_1, y_2)$. Then we say the **base partition sits the same** in the two transformations.*

In this section we show that the base partition in those two skew products does not sit the same. In section 9 we will show how to extend this result to show that the base factors do not sit the same. Now we prove that an isomorphism that is the identity on the base between $B \times_\pi S$ and $B \times_{\pi'} S$ exists if and only if π is conjugate to π' . More generally, an isomorphism that is the identity on the base between $(B \times_\pi S)^j$ and $(B \times_{\pi'} S)^j$ exists if and only if π^j is conjugate to $(\pi')^j$.

To classify the codes to a coordinate i where $(\pi')^m(i) = i$, we will look at codes of the form Ψ such that

$$\Psi : \Omega' \times \Omega \times \Omega \dots \rightarrow \Omega' \times \Omega \text{ and } \Psi(B \times_{\pi} S)^m = (B \times_{id} S)^m \Psi.$$

and Ψ restricted to the base is the identity. Also define

$$\Psi^* : \Omega' \times \Omega \times \Omega \dots \rightarrow \Omega' \times \Omega \rightarrow \Omega$$

so that if $\Psi(b, y_1, y_2, \dots) = (b, x)$ then define $\Psi^*(b, y_1, y_2, \dots) = x$. Our next goal is to prove that if $\Psi^*(b, y_1, y_2, \dots) = x$ then x eventually lines up with one of the y_i . This next lemma is the main tool for converting a counterexample to a relative counterexample over the same base.

Lemma 6.1 *For all $\epsilon > 0$ there exists integers N and L and a set $G, \mu(G) > 1 - \epsilon$ such that for $n > L, n \neq 1 \pmod{3}$, we have the following. If $(b, y_1, y_2, \dots), (b', z_1, z_2, \dots) \in G$, the n overlap around 0 for $y_i, 1 \leq i \leq N$, is the same as the n overlap around 0 for z_i and the n blocks around 0 for y_i and $z_i, 1 \leq i \leq N$, have the same number of f at the beginning of n blocks, then the n block around 0 of $\Psi^*(b, y_1, y_2, \dots)$ lines up with the n block around 0 of $\Psi^*(b', z_1, z_2, \dots)$ to within $6(h(n-1) + S(n))$.*

Proof: Find an N so that a finite approximation, Ψ_N , of Ψ is $\epsilon(\alpha_m)^2/100$ good. Without loss of generality $(1, \dots, N)$ can be taken to be π invariant. Find an $L > N$ so that the ergodic theorem on the set of bad coding has kicked in $\epsilon(\alpha_m)^2/100$ well by time $h(L)$ and almost every point in Ω has $(-3\epsilon_L h(L), 3\epsilon_L h(L))$ in an L block.

Choose $n > L$. Given most any (b, y_1, y_2, \dots) and (b', z_1, z_2, \dots) that code well and that for $i, 1 \leq i \leq N, y_i$ and z_i have n blocks around 0 that line up exactly and have the same number of f at the beginning, we will find a point (b'', w_1, w_2, \dots) that agrees with both (b, y_1, y_2, \dots) and (b', z_1, z_2, \dots) on long regions. Pick a base $n-1$ block for (b, y_1, y_2, \dots) and one for (b', z_1, z_2, \dots) that do not overlap and the points code well over these blocks. Find a base point b'' such that at least $1/3$ of the $n-1$ block for b it matches b perfectly and on at least $1/3$ of the $n-1$ block for b' it matches b' perfectly. For $i, 1 \leq i \leq N$, choose w_i such that the $n-1$ block around 0 lines up exactly with that for y_i and z_i and matches y_i exactly over the $n-1$ block for b and matches z_i exactly over the $n-1$ block for b' . We also want to make sure that (b'', w_1, w_2, \dots) codes well over both base $n-1$ blocks.

Now (b'', w_1, w_2, \dots) agrees with $(b, y_1, \dots, y_n, \dots)$ on a region of length $H(n-1)/3$ and agrees with (b', z_1, z_2, \dots) on a region of length $H(n-1)/3$. Thus we have that $\Psi_N(b'', w_1, w_2, \dots)$ agrees with $\Psi_N(b, y_1, y_2, \dots)$ on a region of length nearly $H(n-1)/3$ and $\Psi_N(b'', w_1, w_2, \dots)$ agrees with $\Psi_N(b', z_1, z_2, \dots)$ on a region of length nearly $H(n-1)/3$. All the names code well over these regions so $\Psi(b'', w_1, w_2, \dots)$ disagrees with $\Psi(b, y_1, y_2, \dots)$ on a fraction less than $\alpha_m/10$ of the times in a region of length nearly $H(n-1)/3$ and $\Psi(b'', w_1, w_2, \dots)$ disagrees with $\Psi(b', z_1, z_2, \dots)$ on a fraction less than $\alpha_m/10$ of the times in a region of length nearly $H(n-1)/3$. Corollary 4.10 tells us a region of length $H(n-1)/3 > \epsilon_n h(n)$ is long enough to tell where an n block is for $n \neq 1 \pmod{3}$. Thus $\Psi(b, y_1, y_2, \dots)$ and $\Psi(b', z_1, z_2, \dots)$ line up to within $3(h(n-1) + S(n))$.

Since this works for most (b, y_1, y_2, \dots) and (b', z_1, z_2, \dots) we can find a point (b, x_1, x_2, \dots) that works well with most (b', z_1, z_2, \dots) that have the same overlap and f . If (b, y_1, y_2, \dots) and (b, z_1, z_2, \dots) are both in the good set then the n blocks of $\Psi^*(b, y_1, y_2, \dots)$ and $\Psi^*(b', z_1, z_2, \dots)$ line up to within $6(h(n-1) + S(n))$. Let G be the union of these good sets for all overlaps and f . \square

Now we show that not only does the n overlap of the skew product names determine the name we are coding to, but if the isomorphism is the identity on the base it does much more. In this case the n block of the name we are coding to lines up with the n block of the of one of the skew product names we are coding from. This next lemma is also important for what it does not prove as what it proves. We need the code Ψ to be over the same base because if we skew two points with n blocks that overlap, but do not line up, over the same base name then their $n-2$ blocks in the overlap line up in almost all possible

ways with almost equal probability. This is not necessarily true if we skew two points over different base points. It is because Ψ is over the same base that we are proving conditions about the base partition sitting the same, not the base factor.

Lemma 6.2 *For any $\epsilon > 0$ there exists an M such that for any $n > M$ and 3 does not divide n , and there exists a set G , $\mu(G) > 1 - \epsilon$ and all $(b, y_1, y_2, \dots) \in G$, the n block of $\Psi^*(b, y_1, y_2, \dots)$ around 0 lines up (to within $6(h(n-1) + S(n))$) with the n block around 0 of y_i for some i , $1 \leq i \leq M$.*

Proof: Choose a point (b, y_1, y_2, \dots) so that no two of the n blocks of the y_i , $1 \leq i \leq N$, most of i in the $n-2$ overlaps have $B \times_\pi S(b, y_1, y_2, \dots) \in G$ from the previous lemma. Choose any overlap of N $n-2$ blocks, A . Since Ψ codes over the same base Lemma 2.2 applies and tells us in the n overlap of y_1, \dots, y_N at the times we see all the $n-2$ blocks that agree within $4n$ with A we see the $n-2$ block of $\Psi^*(b, y_1, y_2, \dots)$ in every position (within $6(h(n-3) + S(n-2))$) with almost equal frequency. There are $(8n+1)^N (n-2)^N$ choices of close overlaps and f , but at least $N(n-2)/6$ choices of the lining up (within $6h(n-3) + S(n-2)$) of the $n-2$ block of z . Since the latter is growing exponentially, it could not be these overlaps close to A that uniquely determine the position of the $n-2$ block of $\Psi^*(b, y_1, y_2, \dots)$ (to within $6h(n-3) + S(n-2)$). But this holds for all A . Thus the $n-2$ block overlaps and f of (b, y_1, y_2, \dots) cannot determine the position of most of the $n-2$ blocks of $\Psi^*(b, y_1, y_2, \dots)$. \square

The reason we were able to study (y_1, y_2, \dots) and $\Psi^*(b, y_1, y_2, \dots)$ instead of (b, y_1, y_2, \dots) and $\Psi(b, y_1, y_2, \dots)$ is because Ψ is the identity restricted to the base.

Lemma 6.3 *There exists an i such that for almost all $(b, y_1, \dots, y_j, \dots)$, the n block of $\Psi^*(b, y_1, y_2, \dots)$ eventually lines up with y_i .*

Proof: Choose n so that $l(n) < 1/2M$ and the previous lemma holds for n . The ergodic theorem and the previous lemma for n large enough we find that most of the n blocks of $\Psi^*(b, y_1, y_2, \dots)$ lines up with an n block of one of the y_i . We will now show that for n large enough it is always the same y_i . Define B_i to be the collection of points (b, y_1, y_2, \dots) that have the n block around 0 of y_i and the n block around 0 of $\Psi^*(b, y_1, y_2, \dots)$ lining up. By the previous lemma $\sum_1^M \mu(B_i) > 1/2$. Now if (b, y_1, y_2, \dots) satisfies the ergodic theorem for the sets B_i the long term averages of (b, y_1, y_2, \dots) are at least $1/2M$ for some B_i . We now wish to show that for this i the n block structures for n large enough $\Psi^*(b, y_1, y_2, \dots)$ and y_i line up. If they didn't then there would be arbitrarily large m such that the m block around 0 of $\Psi^*(b, y_1, y_2, \dots)$ and of y_i didn't line up. Lemma 2.3 implies that for $m > n$ less than $l(n) < 1/2M$ of the n blocks in the m overlap line up. This contradicts the fact that for all sufficiently long stretches at least $1/2M$ of the n blocks line up. Thus any (b, y_1, \dots, y_n) that satisfies the ergodic theorem for the B_i must have $\Psi^*(b, y_1, \dots, y_n)$ that lines up with y_i . \square

Theorem 6.1 *If the base partition sits the same in $(B \times_\pi S)^j$ and $(B \times_{\pi'} S)^j$ then π^j is conjugate to $(\pi')^j$.*

Proof: By the lemma above if we are coding by Φ^{-1} from $(B \times_{\pi'} S)^j$ to $(B \times_\pi S)^j$, Ψ_v pairs up a coordinate v with one coordinate, $\alpha(v)$, that we are coding from. Since for almost every point $(b, y_1, y_2, \dots) \in \Omega^V$ none of the points y_i and y_j have block structures that eventually line up, we know that α is well defined. Since this works for every v the map α is onto. We could also look at the map Φ and to conclude that α is in fact 1-1.

7 Isomorphisms of Factors over the same Base

In this section we will work with a very special collection of factors which are generated by a group of permutations. A group of permutations $P \subset \text{Sym}(V)$ generates a factor, $B \times_\pi S \text{ mod } P$, of $B \times_\pi S$ if for all $p \in P$ there exists $p' \in P$ such that $\pi p = p' \pi$. This factor associates points (b, y_1, y_2, \dots) and

(b, y'_1, y_2, \dots) if $(b, y'_1, y'_2, \dots) = (b, y_{p(1)}, y_{p(2)}, \dots)$ for some $p \in P$. We want to investigate when a factor generated by P of a map $B \times_\pi S$ is isomorphic to a factor generated by P' in a map $B \times_{\pi'} S$. We will deal only with the special case where P is a group generated by permutations p_i , each of which has cycles of only one finite length, which act nontrivially only on disjoint subsets V_i of V . Throughout the rest of the paper we will only deal with factors generated by permutation groups of this form.

If there exists an invertible map $\alpha : V \rightarrow V'$ such that $\alpha(P) = P'$ and $\alpha^{-1}\pi\alpha = \pi' \bmod P'$ it is easy to construct an isomorphism between $B \times_\pi S \bmod P$ and $B \times_{\pi'} S \bmod P'$ which is the identity on the base factor. The existence of such an α is almost certainly a necessary condition as well. However in this section we prove a slightly weaker necessary condition. It should not be too difficult to get the necessary and sufficient condition, or to work with more general permutation groups, but we do not because we do not need to for any of the examples we want to construct and it would even further complicate the notation.

For example, consider the transformation $B \times_{id} S$ on Ω^4 . Define P_1 to be the subgroup of Sym_4 generated by the permutation $(1,2)(3,4)$ and P_2 to be the subgroup generated by the two permutations $(1,2)$ and $(3,4)$. We will show that the base partitions in $B \times_{id} S$, $B \times_{id} S \bmod P_1$, and $B \times_{id} S \bmod P_2$ do not sit the same.

If p is an n cycle on $(1, \dots, n)$ define Ω_p^n to be the space obtained by associating a point $(\omega_1, \dots, \omega_n)$ with the points of the form $(\omega_{p^m(1)}, \dots, \omega_{p^m(n)})$. We also define a partition, Q_p^n , on Ω_p^n . Given a n cycle $p \in \text{Sym}(1, \dots, n)$ we associate $q = (q_1, q_2, \dots, q_n)$ and $q' = (q'_1, q'_2, \dots, q'_n)$ (all $q_i, q'_i \in Q$) if for some m and all i $q_i = q'_{p^m(i)}$. $Q_p^n(y_1, \dots, y_n)$ is the element of Q_p^n that (y_1, \dots, y_n) is in.

In the previous section we took an isomorphism which was the identity on the base and saw how it coded to each coordinate. Now we will look at how our isomorphism codes to $\Omega' \times \Omega_p^n$. Again any isomorphism Φ which is the identity on the base between $(B \times_\pi S \bmod P)^j$ and $(B \times_{\pi'} S \bmod P')^j$ and any multiple m of j give us a plethora of measurable, shift invariant map

$$\Psi : \Omega' \times \Omega \times \dots \rightarrow \Omega' \times \Omega_p^n$$

such that

$$\Psi(B \times_\pi S \bmod P)^m = (B \times_{\pi'} S \bmod P')^m \Psi.$$

So to classify the possible isomorphisms we will study the Ψ . Define Ψ^* so that if $\Psi(b, y_1, y_2, \dots) = (b, x_1, \dots, x_n) \bmod P$ then $\Psi^*(b, y_1, y_2, \dots) = (x_1, \dots, x_n) \bmod P$.

By choosing a large enough N this map, Ψ can be approximated arbitrarily well by a map

$$\Psi_N : \Omega' \times \Omega \times \dots \times \Omega \rightarrow Q_p^n$$

which is measurable in $Q^N \vee \dots \vee K^{-N} Q^N$ and $(1, \dots, N)$. Our goal is to show that if $\Psi(\omega_1, \omega_2, \dots) = (y_1, \dots, y_n) \bmod p \in \Omega_p^n$ then there exists $(\omega_{i_1}, \dots, \omega_{i_n})$ such that ω_{i_j} lines up with y_j for all j .

This section combines the ideas of the previous section and of [6]. We now produce analogs the arguments of the previous section to show that a coded cycle depends only on the block structure and f of the original K names, not on the base name or how they were filled in. The desired result is that if $\Psi^*(b, y_1, y_2, \dots) = (x_1, \dots, x_n) \bmod P$ then for each x_i there is a y_j which eventually lines up with x_i .

Lemma 7.1 *For all $\epsilon > 0$ there exists an N and L such that for $n > L$ and $n \not\equiv 1 \pmod 3$, there is a $G, \mu(G) > 1 - \epsilon$ with the following property. If $(b, y_1, y_2, \dots), (b', z_1, z_2, \dots) \in G$, the n overlap around 0 for y_i $1 \leq i \leq N$ lines up perfectly with the n overlap around 0 for z_i and the n blocks around 0 for y_i and z_i , $1 \leq i \leq N$, have the same number of f at the beginning of n blocks and $\Psi^*(b, y_1, y_2, \dots) = (x_1, \dots, x_n) \bmod p$ and $\Psi^*(b', z_1, z_2, \dots) = (w_1, \dots, w_n) \bmod p$, then for each x_i there exists a w_j such that the n blocks around 0 line up to within $6(h(n-1) + S(n))$.*

Proof: Take a finite approximation, Ψ_N of Ψ which is $\epsilon(\alpha_m)^2/100$ good. Without loss of generality $(1, \dots, N)$ can be taken to be π invariant. Find an $L > N$ so that the ergodic theorem on the set of bad coding has kicked in $\epsilon(\alpha_m)^2/100$ well by time $h(L)$ and almost every point in Ω has $(-3\beta_L h(L), 3\beta_L h(L))$ in an L overlap. Choose an $n > L$ and $n \neq 1 \pmod 3$.

Given (b, y_1, y_2, \dots) and (b', z_1, z_2, \dots) that code $\epsilon(\alpha_m)^2/100$ well and have the same overlap and f we will find a point (b'', w_1, w_2, \dots) that agrees with both (b, y_1, y_2, \dots) and (b', z_1, z_2, \dots) on long stretches. Choose one base $n-1$ block for (b, y_1, y_2, \dots) and one for (b', z_1, z_2, \dots) that code well and don't overlap. Find a base point b'' such that at least $1/3$ of the $n-1$ block for b it matches b perfectly and on at least $1/3$ of the $n-1$ block for b' it matches b' perfectly. For $i, 1 \leq i \leq N$, choose w_i such that the $n-1$ block around 0 lines up exactly with that for y_i and z_i and matches y_i exactly over the $n-1$ block for b and matches z_i exactly over the $n-1$ block for b' . We also want to make sure that (b'', w_1, w_2, \dots) codes well over both base $n-1$ blocks. The number of (b, y_1, y_2, \dots) and (b', z_1, z_2, \dots) such that we cannot find such a (b'', w_1, w_2, \dots) is small.

Now the P and Q^N names of (b'', w_1, \dots, w_N) agrees with (b, y_1, \dots, y_N) on a region of length $H(n-1)/3$ and agrees with (b', z_1, \dots, z_N) on a region of length $H(n-1)/3$. Thus $\Psi_N(b'', w_1, w_2, \dots)$ agrees with $\Psi_N(b, y_1, y_2, \dots)$ on a region of length nearly $H(n-1)/3$ and $\Psi_N(b'', w_1, w_2, \dots)$ agrees with $\Psi_N(b', z_1, z_2, \dots)$ on a region of length nearly $H(n-1)/3$. All the names code well over both of these regions so $\Psi(b'', w_1, w_2, \dots)$ disagrees with $\Psi(b, y_1, y_2, \dots)$ on a fraction less than $\alpha_m/100$ of the times in a region of length nearly $H(n-1)/3$ and $\Psi(b'', w_1, \dots, w_n, \dots)$ disagrees with $\Psi(b', z_1, z_2, \dots)$ on a fraction less than α_n of the times in a region of length nearly $H(n-1)/3$. As a region of length $H(n-1)/3$ is long enough to tell where an $n-1$ block is, $\Psi(b, y_1, y_2, \dots)$ and $\Psi(b', z_1, z_2, \dots)$ both line up with $\Psi(b'', w_1, w_2, \dots)$. Thus they line up to within $6(h(n-1) + S(n))$. Let G be the union of the good sets for all overlaps and choices of f . \square

Again this next lemma is only possible because Ψ is the identity on the base.

Lemma 7.2 *There exists an M such that for all $n > M$ and 3 does not divide n there exists a set $G, \mu(G) > 1 - \epsilon$, such that $(b, y_1, y_2, \dots) \in G$ and $\Psi^*(b, y_1, y_2, \dots) = (z_1, \dots, z_s) \pmod p$ implies that there exist i_1, \dots, i_s such that the n block overlaps of (z_1, \dots, z_s) around 0 lines up with the n block overlaps around 0 of $(y_{i_1}, \dots, y_{i_s})$.*

Proof: Choose a point (b, y_1, y_2, \dots) so that most of the i in $n-2$ overlaps have $B \times_\pi S(b, y_1, y_2, \dots) \in G$ from the previous lemma, and none of the n blocks line up. Choose any overlap of N $n-2$ blocks, A . Because Ψ is over the same base we can apply lemma 4.1 which tells us in the n overlap of y_1, \dots, y_N at the times that we see the $n-2$ blocks that agree within $4n$ with A we see the $n-2$ block of $\Psi^*(b, y_1, y_2, \dots)$ in every position (within $6(h(n-3) + S(n-2))$) with almost equal frequency. There are $(8n+1)^N (n-2)^N$ choices of close overlaps and f , but at least $N(n-2)/6$ choices of the lining up (within $6(h(n-3) + S(n-2))$) of the $n-2$ block of z . Since the latter is growing exponentially in n it could not be these overlaps close to A that uniquely determine the position of the $n-2$ block of $\Psi^*(b, y_1, y_2, \dots)$ (within $6h(n-3) + S(n-2)$). But this holds for all A . Thus the $n-2$ block overlaps and f of (b, y_1, y_2, \dots) cannot determine the position of most of the $n-2$ blocks of $\Psi^*(b, y_1, y_2, \dots)$. \square

Lemma 7.3 *There exists a choice of i_1, \dots, i_s such that for almost every point (b, y_1, y_2) , if we have $\Psi^*(b, y_1, y_2, \dots) = (z_1, \dots, z_s) \pmod p$, then the block overlaps of (z_1, \dots, z_s) around 0 line up with the block overlaps of the $(y_{i_1}, \dots, y_{i_s})$ around 0 for all n large enough.*

Proof: Choose n so that the previous lemma holds for n and $l(n) < 1/2N^s$. Call the points in the collection of overlaps of n blocks that have the n block of y_{i_1}, \dots, y_{i_s} and the n block of $\Psi(y_1, \dots, y_j)$ lining up B_{i_1, \dots, i_s} . The previous lemma tells us $\sum \mu(B_{i_1, \dots, i_s}) > 1/2$. Now if (y_1, y_2, \dots) satisfies the ergodic theorem for the sets B_{i_1, \dots, i_s} the long term averages of (y_1, y_2, \dots) are at least $1/2N^s$ for some B_i . We now wish to show that for this i_1, \dots, i_s the block structures bigger than n of $\Psi^*(b, y_1, \dots, y_j, \dots)$

and $(y_{i_1}, \dots, y_{i_s})$ line up. If they didn't then there would be some m and j such that the m block around 0 of z_j lines up with none of the m blocks around 0 for $(y_{i_1}, \dots, y_{i_s})$. By Lemma 2.3 this implies that less than ϵ of the n blocks for z_j inside the m overlap line up with an n block for some y_{i_k} . This contradicts the fact that for all sufficiently long stretches at least $l(n) < 1/2N^s$ of the n overlaps line up. Thus any points that satisfies the ergodic theorem for the B_{i_1, \dots, i_s} must code to a point that eventually lines up with y_{i_1}, \dots, y_{i_s} . \square

If $P = id, P_1 = (1, 2, 3, 4), P_2 = (1, 2)$, and $V = (1, 2, 3, 4)$, we want to show that none of $B \times_\pi S$ mod $P, B \times_\pi S$ mod P_1 , and $B \times_\pi S$ mod P_2 are isomorphic. This next lemma will do just that. We say that coordinates i and j are **associated** under the factor generated by P if there exists a $p \in P$ such that $p(i) = j$. This divides the coordinates into equivalence classes.

Lemma 7.4 *If we have an isomorphism Φ from $(B \times_\pi S \text{ mod } P)^j$ to $(B \times_{\pi'} S \text{ mod } P')^j$ which is the identity on the base, such that coordinates $(1, \dots, n)$ are associated by P , and they line up with coordinates i_1, \dots, i_n , then i_1, \dots, i_n must be associated by P' .*

Proof: Suppose not. Then there exists a minimal P' invariant subset S' such that $S' \cap (i_1, \dots, i_n) \neq \emptyset$ and $S'^C \cap (i_1, \dots, i_n) \neq \emptyset$. Suppose

$$\Phi((b, x_1, \dots, x_n, \dots) \text{ mod } P) = ((b, y_1, \dots, y_n, \dots) \text{ mod } P').$$

We want to look at $\Psi_{S'}$

$$(b, x_1, x_2, \dots) \rightarrow (b, x_1, x_2, \dots) \text{ mod } P \rightarrow (y_i, \dots)(i \in S') \text{ mod } P'.$$

From our classification of codes we know that there is a set S of $|S'|$ coordinates such that for every $j \in S, x_j$ lines up with some $y_i, i \in S'$ for almost all points (b, x_1, x_2, \dots) . Because $S' \cap (i_1, \dots, i_n) \neq \emptyset$ we have that $S \cap (1, \dots, n) \neq \emptyset$ and because $S'^C \cap (i_1, \dots, i_n) \neq \emptyset$ we have that $S^C \cap (1, \dots, n) \neq \emptyset$.

Now take a permutation $p \in P$ which contains the cycle $(1, \dots, n)$. Since $S \neq (1, \dots, n)$ the points $x_{p(i)}, i \in S$, don't all line up with one of the points y_{i_1}, \dots, y_{i_n} . On the other hand

$$(b, x_{p(1)}, x_{p(2)}, \dots) \rightarrow (b, x_1, x_2, \dots) \text{ mod } P \rightarrow (b, y_1, y_2, \dots) \text{ mod } P'.$$

So the points $x_{p(i)}, i \in S$, should all line up with one of the points y_{i_1}, \dots, y_{i_n} . This is a contradiction. \square

Take $\pi = id$ and $V = (1, 2, 3, 4)$. Let the group P_3 be generated by the cycles $(1,2)$ and $(3,4)$ and let the group P_1 be generated by the permutation $(1,2)(3,4)$. The group P_3 will create a four point factor of $B \times_\pi S$ while P_1 will create a two point factor of $B \times_{\pi'} S$. The last lemma says nothing about whether the base partition sits the same in these two transformations. The purpose of the next lemma is to be able to say that the factor generated by P is not isomorphic to the one generated by P' .

Lemma 7.5 *If*

1. Φ is an isomorphism between $(B \times_\pi S \text{ mod } P)^j$ and $(B \times_{\pi'} S \text{ mod } P')^j$ which is the identity on the base,
2. (i_1, \dots, i_n) and (j_1, \dots, j_n) are linked under P' , and
3. they line up with the cycles $(1, n)$ and $(n+1, 2n)$,

then $(1, n)$ and $(n+1, 2n)$ are linked under P .

Proof: We have Φ' , the factor map composed with the isomorphism

$$\Phi'(b, x_1, x_2, \dots) \rightarrow (b, x_1, x_2, \dots) \text{ mod } P \rightarrow (b, y_1, y_2, \dots) \text{ mod } P'.$$

We define “Pairings” to be all possible ways to divide $i_1, \dots, i_n, j_1, \dots, j_n$ into pairs so no pair is of the form (i_k, i_m) or (j_k, j_m) . Define a function from

$$F : \Omega \times \Omega \times \dots \rightarrow \text{“Pairings”}$$

in the following way. Take (b, x_1, x_2, \dots) and find (b, y_1, y_2, \dots) such that

$$\Phi'(b, x_1, x_2, \dots) = (b, y_1, y_2, \dots) \text{ mod } P.$$

For each k , $1 \leq k \leq n$ write down the pair of coordinates that line up with i_k and j_k . Call those n pairs $F(b, x_1, x_2, \dots)$. This is well defined after we have made our choice of i_1 and j_1 because the cycles (i_1, \dots, i_n) and (j_1, \dots, j_n) are linked under P' .

Choose k so that $\pi^k = (\pi')^k$ when restricted to $(1, \dots, 2n)$ and $(i_1, \dots, i_n, j_1, \dots, j_n)$ respectively. F is $(B \times_{\pi'} S)^k$ invariant because

$$\Phi'(B \times_{\pi'} S)^k(x_1, x_2, \dots) = (B \times_{\pi} S)^k(y_1, y_2, \dots) \text{ mod } P$$

and these points generate the same pairings. Thus we have that F is a shift invariant function into a finite set. Thus F is constant a.e.

Now suppose $(1, n)$ and $(n+1, 2n)$ are not linked. Then there would exist a p such that $p(1) = 2$ and $p(n+1) = n+1$. But

$$(b, x_{p(1)}, \dots, x_{p(n+1)}, \dots) = (b, x_2, \dots, x_{n+1}, \dots) \text{ and } (b, y_1, y_2, \dots)$$

give us different pairings than $(b, x_1, x_2, \dots, x_{n+1}, \dots)$ and (b, y_1, y_2, \dots) . This gives us that

$$F(b, x_{p(1)}, x_{p(2)}, \dots) \neq F(b, x_1, x_2, \dots),$$

which is a contradiction. □

Thus an isomorphism takes n cycles to n cycles and linked n cycles to linked n cycles. We summarize our results in the following lemma. This next lemma applies only to P of the our special form.

Theorem 7.1 *If there exists an isomorphism between $(B \times_{\pi} S \text{ mod } P)^j$ and $(B \times_{\pi'} S \text{ mod } P')^j$ which is the identity on the base then P is conjugate to P' . Also, restricted to the unlinked coordinates, we must have π^j conjugate to $(\pi')^j$. A sufficient condition is there exist an $\alpha : V \rightarrow V'$ such that $\alpha P \alpha^{-1} = P'$ and $\alpha \pi^j \alpha^{-1} = (\pi')^j \text{ mod } P'$.*

Proof: The necessary condition is a summary of Lemmas 7.4 and 7.5 as well as Theorem 6.1. The sufficient condition comes from the map, Ψ_{α} , which takes a point $(b, x_1, x_2, \dots) \text{ mod } P$ to $(b, x_{\alpha(1)}, x_{\alpha(2)}, \dots) \text{ mod } P'$. □

8 Relative Minimal Self Joinings

In the previous two sections we established a number of results concerning the isomorphisms of our skew products, with the K automorphisms, which are the identity on the base. Now we want to get similar results when the skew products are constructed with zero entropy transformations. The specific conditions that we need are

Theorem 8.1 *For any isomorphism between $(B \times_{\pi} S)^j$ and $(B \times_{\pi'} S)^j$ which is the identity on the base there is a map $\alpha : V \rightarrow V'$ such that $\alpha \pi^j \alpha^{-1} = (\pi')^j$.*

Theorem 8.2 *For any groups P and P' there exists an isomorphism between $(B \times_{\pi} S \text{ mod } P)^j$ and $(B \times_{\pi'} S \text{ mod } P')^j$ which is the identity on the base, then there is a map $\alpha : V \rightarrow V'$ such that $\alpha P \alpha^{-1} = P'$ and $\alpha \pi^j \alpha^{-1} = p'(\pi')^j$ for some $p' \in P'$.*

Theorem 8.3 *If $V = 1$ then the base factor in $B \times_{id} S$ relatively prime.*

It is possible to obtain the first two results by copying the previous two sections. The only change that needs to be made is to omit any mention of the number of f at the beginning of an n block. Another way is to classify the possible relative measures over the base factor, as Rudolph did in [20]. This will say that our zero entropy skew products have the relativized version of minimal self joinings. By this we mean that if we put any shift invariant measure on our skew product then the relative measure given the base factor is an off diagonal measure. From this it is possible to classify all of the possible isomorphisms of our skew products that are the identity on the base factor in exactly the same way a Rudolph classified the isomorphisms of a transformation with minimal self joinings.

The condition on a collection of points which let Rudolph classify factors and isomorphisms of a family of transformations was that either two or more points lie on the same orbit or their names are mutually independent. In this section we want to establish the analog of that condition for names in the skew product.

The construction of the base blocks and the zero entropy blocks gave some conditions on the minimum size of the blocks. Now we show that if we choose the blocks to be sufficiently large then we will get a relative analog of minimal self joinings. We want to build our base names so that most n blocks have the following property. If a base n block is covered by a good $n + 1$ block overlap then for any $n - 1$ block overlap, (i_1, \dots, i_k) and base $n - 2$ block A

$$\left| \begin{array}{l} (\# \text{ of } t, 0 < t \leq H(n) \mid T^{f(t)}(\omega_1) \text{ is in the } i_1 \text{th position of an } n - 1 \text{ block,} \\ T^{f(t)}(\omega_2) \text{ is in the } i_2 \text{th position of an } n - 1 \text{ block, } \dots \\ T^{f(t)}(\omega_k) \text{ is in the } i_k \text{th position of an } n - 1 \text{ block and} \\ B^t(b) \text{ is in the first position of } n - 2 \text{ block } A) \mid / \\ (\# \text{ of } t, 0 < t \leq H(n) \mid B^t(b) \text{ is in the first position of } n - 2 \text{ block } A_n) \\ -(1/h(n - 1))^k \mid < \epsilon_n. \end{array} \right.$$

where $f(t) = \sum_0^{t-1} \chi_0(B^t(b))$ and $f(0) = 0$.

In order to get this property we will have to alternate between choosing parameters for the zero entropy blocks with parameters for the base blocks. Since the construction of the zero entropy and base blocks only gave us a minimum block size necessary, and the new constraints we are going to add will only make blocks longer the construction is possible.

Getting this property requires choosing $M(n)$ for the base and $N(n + 1)$ for the zero entropy transformation to be sufficiently large. By Corollary 2.1 we have an infinite pseudorandom sequence $a_{i,n}$ and an N_0 such that for $N > N_0$, $a_{1,n}, \dots, a_{N,n}$ is a good pseudorandom sequence. We will keep increasing both $M(n)$ for the base and $N(n + 1)$ for the zero entropy until the property above is satisfied.

Lemma 8.1 *There exists a choice of $M(n)$ and $N(n + 1)$ such that the above condition is satisfied.*

Proof: Pick a value for $M(n)$ large enough so that most base n blocks contain every base $n - 1$ block nearly the same number of times. Based on this choose $N(n + 1)$ so that $2n^2\epsilon_n \geq H(n)/h(n + 1) \geq n^2\epsilon_n$. Now pick one good zero entropy $n + 1$ overlap, $(j_1, \dots, j_n), 0 \geq j_i \geq -h(n + 1)$, one zero entropy $n - 1$ overlap, $(k_1, \dots, k_n), 0 \geq k_i \geq -h(n - 1)$, and one base $n - 2$ block, A . Thus an $n + 1$ overlap is good if it is long enough, all $j_i \geq -h(n + 1)(1 - \epsilon_n)$ and $\min |j_i - j_k| > h(n) + S(n + 1)$. Consider all of the possible ways to fill in $M(n)$ $n - 1$ blocks to form an n block.

By an $n + 1$ block overlap, (j_1, \dots, j_n) over an n block we mean that above the first symbol in our base n block the first skewed name is in the j_1 position in the $n + 1$ block, the second skewed name is in the j_2 position in the $n + 1$ block, etc. Because the number of zero in each n block is the same, if we have a fixed $n + 1$ block overlap, (j_1, \dots, j_n) , and a fixed number of 0s and 1s at the start of the n block then the $n + 1$ block overlap over the i th $n - 1$ block inside the n block is determined.

Define the function F_i on the set of all $n - 1$ base blocks so that $F(B)$ is the number of times that base $n - 2$ block A occurs in B and the zero entropy $n - 1$ overlap over the first symbol in block A is (k_1, \dots, k_n) if B is the i th $n - 1$ block in a base n block.

The F_i are uniformly bounded by the number of $n - 2$ blocks in an $n - 1$ block. Since the F_i are defined on a finite set and take only a finite number of values, there are only finitely many possible distributions of the F_i . Since the choice of $n - 1$ blocks is independent, the F_i are independent random variables. Thus the central limit theorem applies. So as $M(n)$ increases, the percentage of all of the n blocks, $C = (B_1, \dots, B_{M(n)})$, with a fixed number of 0 at the beginning for which $|\frac{\sum_1^{M(n)} F_i(B)}{M(n)} - \text{Expectation}(\sum F_i/M(n))| > \epsilon$ decreases exponentially as $M(n) \rightarrow \infty$. This exponential rate is independent of all of our choices. As $M(n)$ and $N(n + 1)$ increase the number of choices of $n + 1$ overlaps, (j_1, \dots, j_n) increase only polynomially while the number of $n - 2$ base blocks, A , $n - 1$ overlaps, (k_1, \dots, k_n) , and choices of 0s and 1s remain constant. Thus the percentage of n base blocks such that for some $n + 1$ overlap, $n - 1$ overlap, and $n - 2$ base block, above the $n - 2$ base block the $n - 2$ overlap is seen with the wrong frequency is going to zero as $M(n)$ and $N(n + 1)$ are increasing to ∞ . Thus such a choice of $M(n)$ and $N(n + 1)$ exist. \square

Once this condition is established it is straightforward to prove the relative analog of Rudolph's results.

9 Isomorphisms over any Base

In this section we want to show that if our base factor B sits the same under the action of two skew products, $B \times_\pi S$ and $B \times_{\pi'} S$, then there exists α such that $\alpha\pi\alpha^{-1} = \pi'$. Furthermore if we look at an isomorphism of factors given by permutations p and p' (or permutation groups P and P'), then conjugation by α takes p into p' (or P into P').

The results that we have from the previous sections are that

1. For any isomorphism between $(B \times_\pi S)^j$ and $(B \times_{\pi'} S)^j$ which is the identity on the base factor, there is a map α such that $\alpha(\pi)^j\alpha^{-1} = (\pi')^j$.
2. For any groups (of the appropriate form) P and P' , and isomorphism between $(B \times_\pi S \text{ mod } P)^j$ and $(B \times_{\pi'} S \text{ mod } P')^j$ which is the identity on the base factor there is a map $\alpha : V \rightarrow V'$ such that $\alpha P\alpha^{-1} = P'$ and restricted to unassociated coordinates $\alpha\pi^j\alpha^{-1} = (\pi')^j$.

We want to establish the same two statements for isomorphisms over any base. First consider the case where $P = \text{id}$. Suppose there exists an isomorphism Φ between $B \times_\pi S$ that preserves the base factor F and $B \times_{\pi'} S$ and

$$\Phi(b, y_1, y_2, y_3, \dots) = (b', y'_1, y'_2, y_3, \dots).$$

If f is a permutation of V that moves only finitely many elements, what can we say about

$$\Phi(b, y_{f(1)}, y_{f(2)}, y_{f(3)}, \dots) = (b'', y''_1, y''_2, y''_3, \dots)?$$

Since Φ preserves the base factor the base coordinate be B'' must be the same as B' . Because f moves only finitely many coordinates we can find V_i and n_i such that $f|_{V \setminus V'_i} = \text{id}$ and $\pi^{n_i}|_{V_i} = \text{id}$. If this is the case then $\Phi f \Phi^{-1}$ commutes with $(B \times_\pi S)^{n_i}$. Now we are in a position to use the results we have proven about isomorphisms over the same base. Thus every y'_i must eventually line up with a y'_j . Thus by starting with a finite permutation, f , of the y_i we have induced a permutation, $P(f)$, (up to eventually lining up) of the y'_i , which has only finite cycles. For some n $(P(f))^n = \text{id}$ since $(\Phi f \Phi^{-1})^n = \text{id}$.

Now if g is a permutation of V which has only finite cycles but moves infinitely many elements of V , we will get a permutation $P(g)$ by approximating g by permutations which move only finitely many elements of V . To determine $P(g)(i)$, where $(\pi')^n = i$, we find a set of coordinates W such that we have

an $\alpha_n/100$ good map approximating how Φ codes to the i th coordinate. Find g_i that move only finitely many coordinates so that $g_i(j) = g(j)$ for all j on the same π cycle of an element of W . Now for almost all (b, y_1, y_2, \dots) , the i th component of $\Phi g \Phi^{-1}(b, y_1, y_2, \dots)$ lines up with the i th component of $\Phi g_i \Phi^{-1}$. Thus we define $P(g)(i)$ to be $P(g_i)(i)$. This permutation will have finite cycles.

We now wish to say that this permutation, $P(f)$, is independent of the choice of b and the y_i s. To do this consider the function $C_{\Phi, f, i}(b, y_1, y_2, \dots) =$ the coordinate that y_i lines up with under $\Phi f \Phi^{-1}$. For some high power of $B \times_{\pi'} S$ this is a shift invariant function into a countable set. Thus it is constant a.e. Since there are only a countable number of i , there exists a set of measure one on which all of the $C_{\Phi, f, i}(b, y_1, y_2, \dots)$ are constant. Thus f is mapped to $P(f)$, the permutation that sends i to $C_{\Phi, f, i}$.

We will use the notation $FP(V)$ to represent the subset of $\text{Sym}(V)$, the group of permutations of V , which have only finite cycles. This map P from $FP(V)$ to $FP(V')$ is one to one and onto because the inverse map exists. P also preserves composition. If $f, g, fg \in FP(V)$ then $P(f)P(g) = P(fg)$ because $\Phi f \Phi^{-1} \Phi g \Phi^{-1} = \Phi fg \Phi^{-1}$.

Lemma 9.1 *If $|V| \neq 6$ then P extends to an isomorphism of $\text{Sym}(V)$ and $\text{Sym}(V')$. Thus P is conjugation by a permutation α .*

Proof: This is a simple consequence of the proof in [7] that the only automorphisms of symmetric groups are inner automorphisms, except for one of $\text{Sym}(6)$. \square

Theorem 9.1 *The base factor F sits the same in $B \times_{\pi} S$ as it does in $B \times_{\pi'} S$ if and only if there exists an α such that $\alpha \pi \alpha^{-1} = \pi'$.*

Proof: This is immediate from the previous lemma. \square

Now we want to do a similar analysis for isomorphisms of factor maps. We previously defined two coordinates, i and j , to be linked if for all $p \in P$, $p(i) \neq i$ if and only if $p(j) \neq j$. This association partitions the coordinates into equivalence classes, $V = \cup E_i$. We say that two equivalence classes are similar if $|E_i| = |E_j|$ and $P|_{E_i} = P|_{E_j}$. This partitions the equivalence classes into equivalence classes F_j . The space on which $B \times_{\pi} S \text{ mod } P$ operates is

$$\Omega' \times \Omega_{p_1}^{|E_1|} \times \dots \times \Omega_{p_n}^{|E_n|} \dots$$

where p_n is a generator of $P|_{E_n}$.

In the previous argument we picked a point, saw where it came from under the isomorphism, Φ^{-1} , permuted the coordinates and saw where the new point went under Φ . This generated a permutation. Now we are going to pick a point, (b, y_1, y_2, \dots) each $y_i \in \Omega_{p_i}^{|E_i|}$ and for each F_j a permutation, t_j , of the equivalence classes that make up F_j . These t_j combine to form a permutation, T of V . If T moves only finitely many elements of V then we compare $\Phi(b, y_{T(1)}, y_{T(2)}, \dots)$ and $\Phi(b, y_1, y_2, \dots)$. In a similar manner as before this will generate the direct product of permutations, $P(T)$, with only finite cycles. We then extend this to all permutations that have only finite cycles. By the same argument as before this direct product of permutations is independent of the choice of (x, y_1, y_2, \dots) and preserves compositions. The next lemma says that this can only come from permuting the sets and then permuting the elements of the sets.

Lemma 9.2 *Let $S_i, i \in I$, and $T_j, j \in J$ be a finite or countable number of finite or countable sets such that $|S_i|, |T_j| \neq 6$, $G = \prod \text{Sym}(S_i)$ and $G^* = \prod \text{Sym}(T_j)$, and $FP(G)$ and $FP(G^*)$ the subsets of G and G^* that have only cycles of finite length. If $P : FP(G) \rightarrow FP(G^*)$ is invertible and preserves composition, then Φ extends to an isomorphism of G to G^* . Also there exists a permutation $\sigma : I \rightarrow J$ such that $|S_i| = |T_{\sigma(i)}|$ and $1 - 1$, onto maps $\sigma_i : S_i \rightarrow T_{\sigma(i)}$ such that Φ is simply the isomorphism generated by the composition of σ and the σ_i .*

Proof: This too is a simple extension of the proof in [7]. \square

This lets us conclude all we need to know about isomorphisms over the same base to construct all of our examples.

Corollary 9.1 *If the base factors in $B \times_{\pi} S \bmod P$ and $B \times_{\pi'} S \bmod P'$ sit the same then there must be a pairing of the equivalence classes of similar linked sets of coordinates F_i and F'_i such that each set in a pair is of the same cardinality.*

Corollary 9.2 *If the base factors in $B \times_{\pi} S \bmod P$ and $B \times_{\pi'} S \bmod P'$ sit the same and P and P' are such that the only countable set of similar equivalence classes, T and T' , come from the unassociated coordinates then we have that $\pi|_T$ is conjugate to $\pi'|_{T'}$.*

Proof: By the previous lemma we have $\sigma : T \rightarrow T'$ which is $B \times_{\pi} S$ invariant. Thus $\pi\sigma\pi^{-1}|_T = \pi'|_{T'}$. \square

10 Examples

Now we will use the results we proved in the last section to create a set of relative counterexamples. All of the examples in this section are analogs of zero entropy transformations created by Rudolph in [20]. All of the relatively K counterexamples are in infinite entropy Bernoulli shifts. In the next section we show how to create the counterexamples in finite entropy Bernoulli shifts. W always represents the generating σ -algebra and B represents the base factor of $B \times_{\pi} S$. Although we state our results mentioning two factors in the same Bernoulli shift we construct our examples with factors in different shifts. The two Bernoulli shifts are isomorphic [8]. This allows us to take the isomorphic image of one factor inside the other transformation.

Example 10.1 *Every Bernoulli shift contains an uncountable collection of relatively K (relatively zero entropy) factors which do not sit the same.*

By a different approach Swanson proved in [21] that every Bernoulli shift has an uncountable collection of factors in which the base partition does not sit the same. Let $V = \mathbf{Z}$. Call the j th prime a_j . For each sequence x_i of 0s and 1s we have a permutation π_x of V that has a cycle of length a_j if and only if $x_j = 1$. Any two different sequences x and y will generate permutations which are not conjugate and thus the base factor in $B \times_{\pi_x} S$ will not sit the same as the base factor in $B \times_{\pi_y} S$.

Example 10.2 *Every Bernoulli shift contains a factor which is relatively prime.*

Take $B \times_{id} S$ with $V = (1)$. This is a restatement of Theorem 8.3.

Example 10.3 *Every Bernoulli shift contains two relatively K (relatively zero entropy) factors which do not sit the same under $B \times_{\pi} S$, but sit the same under $(B \times_{\pi} S)^n$ for all $n \neq 1, -1$.*

Take $V = \mathbf{Z}$ and π to be a permutation which for every j , such that j is the product of an odd number of primes, has infinitely many cycles of length j and no cycles with length k , where k is the product of an even number of primes. Take π' to be a permutation such that for every k , k is the product of an even number of primes, there are infinitely many cycles of length k and there are no cycles with length j , where j is the product of an odd number of primes. Since these two permutations do not have the same number of cycles of each length they are not conjugate. Thus B does not sit the same under $B \times_{\pi} S$ and $B \times_{\pi'} S$. However for every $n > 1$ π^n is conjugate to $(\pi')^n$ and therefore B does sit the same under $(B \times_{\pi} S)^n$ and $(B \times_{\pi'} S)^n$.

Example 10.4 *Every Bernoulli shift contains a relatively K (relatively zero entropy) factor with no roots. That is no root of $B \times_{\pi} S$ preserves B .*

Take $V = \mathbf{Z}$ and π to be a permutation with no roots. If R is a root of $B \times_{\pi} S$ preserved F then it would generate a permutation σ . $R^n = B \times_{\pi} S$ would imply $\sigma^n = \pi$. But this is a contradiction.

Example 10.5 *Every Bernoulli shift contains a factor B which has two extensions E_1 and E_2 , such that E_1 and E_2 have no common factors except those contained in B . Yet there exists a factor F which extends both E_1 and E_2 , but contains no factors extending B in which B sits the same as $E_1 \times_B E_2$. This is a counterexample to a relativized version of Furstenberg's disjointness conjecture [3].*

Take $V = \{1, 2\}$ and $B \times_{id} S$. It has a factor which associates points of the form (b, x, y) with (b, x, z) and a second factor which associates (b, x, y) with (b, y, x) . These two factors have no common factor which is strictly larger than the base factor. However $B \times_{id} S$ with $V = \{1, 2\}$ has no factor isomorphic to $B \times_{id} S \bmod (2, 3)$ on $V = \{1, 2, 3\}$

Example 10.6 *Every Bernoulli shift contains two relatively K (relatively zero entropy) factors which weakly sit the same, but do not sit the same. That means there exists two factors B and B' such that B and B' do not sit the same, but W contains a factor F such that B sits the same in F as B' does in W and W contains a factor F' such that B' sits the same in F' as B does in W .*

We will give two different examples of factors weakly sitting the same but not sitting the same. The first is the simplest. The second one is more complicated but can be used to get to the factors to be of relatively finite entropy.

A) Take $\pi = id$ and $V = \mathbf{Z}$. B is the base factor and B' is the isomorphic image of the factor generated by the base in the two point factor obtained by associating points of the form $(b, t_1, t_2, t_3, \dots)$ with $(b, t_2, t_1, t_3, \dots)$ (i.e. $P = (1, 2)$). One factor map is clear and the other takes $(b, t_2, t_1, t_3, t_4, t_5, \dots)$ to $(b, t_3, t_4, t_5, \dots)$. They do not sit the same because one has two coordinates associated but the other does not.

B) Take $V = \mathbf{Z} \times \mathbf{Z}$ and $\pi = id$. P will be generated by one permutation p_i on the coordinates $(i, *)$ for every $i > 0$. Call the j th prime a_j . The permutation p_1 will be $(1, 2)(3, 4)(5, 6)(7, 8) \dots$ and $p_2 = (1, 3, 5, 2, 4, 6)(7, 8, 9, 10, 11, 12) \dots$. Thus $p_2^2 = p_1$. Inductively define $p_n^{a_n} = p_{n-1}$. P' will be the group generated by P and p_1 acting on $(0, *)$.

Since P' is an extension of P the factor generated by P' is clearly a factor generated by P . The factor generated by P is a factor of the factor generated by P' by the following map. Given a point $(b, x_{i,j}, \dots) \bmod P$ map it to $(b, x_{i-1,j}, \dots) \bmod P'$. The two factors do not sit the same because P is generated by one permutation with order 2 while P' is generated by 2 permutations of order 2.

Example 10.7 *Every Bernoulli shift contains an uncountable collection of two point factors in which B does not sit the same. The factors can be either relatively K or relatively zero entropy extensions of B .*

Take π to be a permutation which has two cycles of every prime length. Take a sequence, s_i , of 0s and 1s indexed by the primes. For each sequence we get a permutation p which interchanges the two cycles of length the i th prime if $s_i = 1$ and is the identity on those two cycles if $s_i = 0$. As p has order 2 it generates a 2 point factor. If we take two different sequences of 0s and 1s then they differ for some i . Thus π restricted to the unassociated coordinates of one factor has 2 cycles of length the i th prime while the other factor does not. Thus any two different sequences generate factors in which the base does not sit the same.

Definition 10.1 *Suppose (X, T, W, μ) and (X', T', W', μ') are two transformations, F is a factor of W , B is a factor of F , F' is a factor in W' , and B' is a factor in F' . We say that B, F and W triply sit the same as B', F' and W' if there exists an isomorphism $\Phi : X \rightarrow X'$ such that $\Phi(B) = B'$ and $\Phi(F) = F'$.*

Example 10.8 Every Bernoulli shift contains a countable collection of two point factors F_t and a factor B in which no two of the triples B, F_t, W and B, F_s, W triply sit the same. The factors can be either relatively K or relatively zero entropy extensions of B .

We will construct this example by constructing a group of permutations which has countably many extensions of order 2 which are conjugate but no conjugation leaves the group fixed. Let $\pi = id$ and

$$V = (\mathbf{Z} \times (\cup_1^\infty [a(1, n), \dots, a(2n, n)])) \cup [b_1, b_2, \dots].$$

Define $p(i, n)$ to be

$$((i, a(1, 1)), (i, a(i, n+1))), ((i, a(1, 2)), (i, a(i, n+2))), \dots ((i, a(1, n)), (i, a(i, 2n))).$$

and p_t to be $(b_1, b_{t+1}), (b_2, b_{t+2}), \dots, (b_t, b_{2t})$. For every integer k there are infinitely many $p(i, n)$ that are k linked two cycles. Let our group P be the group generated by all the $p(i, n)$. The extension P_t is generated by P and p_t . All the P_t are conjugate because for every integer k there are infinitely many generators that are k linked two cycles, thus they are all conjugate to P . No conjugation between P_i and P_j can preserve P because p_i and p_j move a different number of the b_k . Now for any t $B \times_{id} S \bmod P_t$ is a 2 point factor in $B \times_{id} S \bmod P$. The base factor in $B \times_{id} S \bmod P_i$ sits the same in $B \times_{id} S \bmod P_j$ since P_i and P_j are conjugate. They do not triply sit the same because no conjugation preserves P .

11 Finite Entropy Examples

In this section we show that we can construct all of our all of our relatively K counterexamples from the previous section with finite entropy Bernoulli shifts. All of the relatively zero entropy counterexamples are already in finite entropy transformations. In order to do this we need to use a relativized version of the weak Pinsker property (WPP).

Lemma 11.1 For any $\epsilon > 0$ there is a factor F which is an extension of the base factor and has less than ϵ more entropy than the base factor.

Proof: Take the factor F_n which associates two points (b, x) and (b, y) if x and y have the same n block structure. That is $K^i(x)$ is in an n block if and only if $K^i(y)$ is in an n block. For n large enough F_n has arbitrarily little more entropy than the base factor. We will show that $B \times_{id} S$ is relatively very weak Bernoulli with respect to each F_n .

The matching to show that $B \times_\pi S$ is very weak Bernoulli with respect to F_n is simple. Take any point in F_n and any two pasts given F_n . The pasts may tell us what the first n block after time 0 is, but they tell us nothing about any subsequent n blocks. Thus we are free to match the futures conditioned on pasts P_1 and P_2 in exactly the same way. \square

We will now show that whatever properties our Bernoulli shifts had relative to the base factors, there exist related factors of finite entropy with the same properties relative to the base factor.

If we had $B \times_\pi S$ and $B \times_{\pi'} S$ so that the base factors do not sit the same then the factors we just described do not sit the same. If they did we could extend this map with an isomorphism of the Bernoulli compliments to get that the base factor in $B \times_\pi S$ sits the same as the base factor in $B \times_{\pi'} S$. Thus the only thing we need to show to get all of our examples for finite entropy extensions is to show that the appropriate maps still exist. We will show how it works for an isomorphism which is the identity on the base. All the other cases are identical.

Theorem 11.1 All of our counterexamples can be constructed with finite entropy transformations.

Proof: Group the set of coordinates V into equivalence classes so that any two coordinates in the same cycle in π or which are associated under P are in the same equivalence class. Choose a function $e(v)$ which is constant on equivalence classes and $\sum_V e(v) < \infty$. For each equivalence class choose an n such that the factor, F_n which associates any two points that have the same n block structure has entropy less than $e(v)$. Call this n , $f(v)$. Then by making the proper associations on each coordinate we get a factor which has an independent Bernoulli compliment and is a finite extension of our base factor.

We have two isomorphic transformations with generating σ algebras W and W' which both have the base factors B and B' . B sits the same in W as B' does in W' . This is done by a map Φ_α

$$\Phi_\alpha(b, y_1, y_2, \dots) \bmod P = (b, y_{\alpha(1)}, \dots, y_{\alpha(n)}, \dots) \bmod P'.$$

The paragraph above constructed factors F and F' which are finite entropy extensions of B and B' . We want to show that B sits the same in F as B' does in F' .

Define $g(v) = g'(\alpha(v)) = \max f(v), f(\alpha(v))$. The functions g and g' define factors G and G' which split off with a Bernoulli compliment. This is done by associating points (b, y_1, y_2, \dots) and (b, z_1, z_2, \dots) if for all v and all $n > g(v)$, the n block structure of y_v and z_v are the same. Thus F can be written as $G \times H$ and F' can be written as $G' \times H'$. The map Φ_α takes G to G' . Since H and H' are Bernoulli shifts of the same entropy there exists an isomorphism which takes H to H' . Combining these two isomorphisms shows that B sits the same in F as B' does in F' . \square

References

- [1] Clark, Jack, A Kolmogorov Shift with no Roots. Ph.D. Dissertation Stanford University (1972).
- [2] Feldman, Jacob, New K -automorphisms and a problem of Kakutani. *Israel J. Math.* **24** (1976), no. 1, 16–38.
- [3] Furstenberg, Harry, Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. *Math. Systems Theory* **1** (1967), 1–49.
- [4] Gerber, Marlies, A zero-entropy mixing transformation whose product with itself is loosely Bernoulli. *Israel J. Math.* **38** (1981),
- [5] Hoffman, Christopher. A loosely Bernoulli counterexample machine. submitted to *Israel J. Math.* (1997).
- [6] Hoffman, Christopher. A K counterexample machine. submitted to *Transactions of the AMS* (1997).
- [7] Maurer, Helmut, Zur Automorphismengruppe der symmetrischen Gruppen. *Die Mathematische Gesellschaft* **11** (1983) no. 2, 265-266.
- [8] Ornstein, Donald, Two Bernoulli shifts with infinite entropy are isomorphic. *Advances in Math.* **5** (1970), 339–348.
- [9] Ornstein, Donald, Factors of Bernoulli shifts are Bernoulli shifts. *Advances in Math.* **5** (1970), 349–364.
- [10] Ornstein, Donald, Bernoulli shifts with the same entropy are isomorphic. *Advances in Math.* **4** (1970), 337–352.
- [11] Ornstein, Donald, *Ergodic Theory, Randomness and Dynamical Systems*, Yale University Press, New Haven, (1974).
- [12] Ornstein, Donald; Shields, Paul, An uncountable family of K -automorphisms. *Advances in Math.* **10**, (1973), 63–88.

- [13] Ornstein, Donald; Rudolph, Daniel; Weiss, Benjamin Equivalence of measure preserving transformations. *Mem. Amer. Math. Soc.* **37** (1982), no. 262.
- [14] Ornstein, Donald, Factors of Bernoulli shifts. *Israel J. Math.* **21** (1975), no. 2-3, 145–153.
- [15] Rahe, M. Relatively finitely determined implies relatively very weak Bernoulli. *Can. J. Math.* Vol. XXX (1978), no. 3, 531-548
- [16] Rudolph, Daniel, Two nonisomorphic K -automorphisms all of whose powers beyond one are isomorphic. *Israel J. Math.* **27** (1977), no. 3–4, 277–298.
- [17] Rudolph, Daniel, Classifying the isometric extensions of a Bernoulli shift. *J. Analyse Math.* **34** (1978), 36-60.
- [18] Rudolph, Daniel, Two nonisomorphic K -automorphisms with isomorphic squares. *Israel J. Math.* **23** (1976), no. 3-4, 274–287.
- [19] Rudolph, Daniel, Counting the relatively finite factors of a Bernoulli shift. *Israel J. Math.* **30** (1978), no. 3, 255–263.
- [20] Rudolph, Daniel, An example of a measure preserving map with minimal self-joinings, and applications. *J. Analyse Math.* **35** (1979), 97–122.
- [21] Swanson, Laif, Generators of factors of Bernoulli shifts. *Pacific J. Math.* **71** (1977), no. 1, 213–220.
- [22] Thouvenot, Jean-Paul, Quelques propriétés des systèmes dynamiques qui se décomposent en un produit de deux systèmes dont l'un est un schéma de Bernoulli. *Israel J. Math.* **21** (1975), no. 2-3, 177–207.