AN ENDOMORPHISM WHOSE SQUARE IS BERNOULLI

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Abstract. One of the corollaries of Ornstein's isomorphism theorem is that if (Y,S,ν) is an invertible measure preserving transformation and (Y,S^2,ν) is isomorphic to a Bernoulli shift then (Y,S,ν) is isomorphic to a Bernoulli shift. In this paper we show that noninvertible transformations do not share this property. We do this by exhibiting a uniformly 2-1 endomorphism (X,σ,μ) which is not isomorphic to the one sided Bernoulli 2 shift. However (X,T^2,μ) is isomorphic to the one sided Bernoulli 4 shift.

1. Introduction

One of the most important questions in ergodic theory is determining when two systems are (measurably) isomorphic. Some of the most studied systems are Bernoulli shifts. The one sided Bernoulli d shift acts on the space $X_d = \{0,1,\dots,d-1\}^{\mathbb{N}}$. The measure μ_d on X_d is defined by

$$\mu_d\{x:\ x_0 = a_0, x_1 = a_1, \dots, x_m = a_m\} = d^{-m}$$

for any $a_0,\dots,a_m \in \{0,1,\dots,d-1\}$. The action $\sigma_d : X_d \to X_d$ is the left shift $\sigma_d(x)_i = x_{i+1}$. The (invertible) Bernoulli d shift acts on the state space $\hat{X}_d = \{0,1,\dots,d-1\}^{\mathbb{Z}}$. The measure $\hat{\mu}_d$ on $\hat{X}_d$ and action $\hat{\sigma}_d : \hat{X}_d \to \hat{X}_d$ are defined in an analogous manner.

It is easy to check that (X_2,σ_2^2,μ_2) is isomorphic to (X_4,σ_4,μ_4). By composing isomorphisms we can see that for any system (Y,S,ν) which is isomorphic to (X_2,σ_2,μ_2), the system (Y,S^2,ν) is isomorphic to (X_4,σ_4,μ_4). In this paper we show that the converse is not true. That is we construct an endomorphism (Y,S,ν) such that (Y,S^2,ν) is isomorphic to (X_4,σ_4,μ_4), but (Y,S,ν) is not isomorphic to (X_2,σ_2,μ_2). This result is in stark contrast with Ornstein's theory on invertible transformations [4]. One result in this theory is that if an invertible transformation (Y,S,ν) is such that (Y,S^2,ν) isomorphic to $(\hat{X}_4,\hat{\sigma}_4,\hat{\mu}_4)$, then (Y,S,ν) is isomorphic to $(\hat{X}_2,\hat{\sigma}_2,\hat{\mu}_2)$ [5].

The main result that we use is due to Hoffman and Rudolph. They introduced a criteria which characterizes when an endomorphism is isomorphic to the one sided Bernoulli d shift [2]. This condition, called tree very weak Bernoulli (t.v.w.B), has many similarities with Ornstein's very weak Bernoulli property, which characterizes when an invertible map is

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isomorphic to a Bernoulli shift [5] [7]. In this sense the work of [2] is similar to other theories, including Feldman's theory of Kakutani equivalence [1], [6] and Rudolph's restricted orbit equivalences [3], that also parallel Ornstein's isomorphism theory.

This paper shows that although the theory of one sided Bernoulli shifts has some strong similarities to the theory of invertible Bernoulli shifts, it also has some major differences. It also shows that the theory of Bernoulli endomorphisms is differs from Kakutani equivalence and restricted orbit equivalences, which both parallel Ornstein's theorem very closely.

In Section 2 we will lay out the notation necessary for describing the tree very week Bernoulli condition as well as for the construction of our endomorphism. In Section 3 we describe the basics of the construction of the endomorphism, leaving some technical details to Section 4. Then in Section 5 we show that (X, σ^2, μ) is isomorphic to the one sided Bernoulli four shift. Finally in Section 6 we show that the endomorphism (X, σ, μ) is not isomorphic to the one sided Bernoulli two shift.

2. Notation

We begin to introduce some notation to help understand the tree structure of a d-adic endomorphism. Consider a rooted d-ary tree with d^n vertices at distance n from the root for each $n \geq 0$. Each vertex at distance n connects to d vertices at the distance $n+1$. For each set of d vertices which connect to the same vertex at one level higher we label them $0, ..., d-1$. Then we give a new label to each vertex other than the root by the sequence of values we see moving from the root to the given vertex. Call this labeled tree $\mathcal{T} = \mathcal{T}^d$. If we truncate the tree at distance $n > 0$ we call it $\mathcal{T}_n$.

We also use the notation $\mathcal{T}$ to refer to the set of vertices of $\mathcal{T}$ and $\mathcal{T}_n$ to refer to the set of vertices of $\mathcal{T}_n$. For $v \in \mathcal{T}$ and at distance i (i.e. $v \in \mathcal{T}_i \setminus \mathcal{T}_{i-1}$) we write $|v| = i$ and we write v as a list of values $v_1, \ldots, v_i$ in $\{0, ..., d-1\}$ where this is the list of labels of the vertices along the branch from the root to v. In this form we can concatenate vertices v' and v by concatenating their labels to form

$$vv' = (v_1, v_2, \ldots, v_{|v|}, v'_1, \ldots, v'_{|v'|}).$$

We say that v' is an **extension** of v if $v' = vv''$ for some $|v''| \geq 1$. We also say that v is a **contraction** of v'.

A $\mathcal{T}, P$ name h is any function from $\mathcal{T} \setminus \{0\}$ to P. If T' is any subtree of $\mathcal{T}$ then we also write T' for the set of vertices of T'. A T', P name h is any function from $T' \setminus \{0\}$ to P. A $\mathcal{T}, P$ name h is **tree adapted** if for all $v \in \mathcal{T}$ and all $0 \leq i < j \leq d-1$ we have $h(vi) \neq h(vj)$. We say that a vertex v is in the **bottom** of T' ($b(T')$) if no extension of v is a vertex in T'. Given any two trees T' and T'' and vertex v which is in the bottom of T' we define T''', the tree with T'' **attached to v** as follows. Let

$$T''' = T' \cup_{v' \in T''} vv'.$$

We define $\hat{T} = T' \circ T''$, the **concatenation** of two trees T' and T'' to be the tree with T'' attached to every vertex which is in the bottom of T'. Thus

$$\hat{T} = T' \cup_{v' \in b(T')} \left(\cup_{v'' \in T''} v'v'' \right).$$

We attach and concatenate tree names in an analogous manner.

Let $\mathcal{A}$ be the collection of all bijections of $\mathcal{T}$ that preserve the tree structure. We refer to this as the group of **tree automorphisms**. Let $\mathcal{A}_n$ be the bijections of $\mathcal{T}_n$ preserving the tree structure. To give a representation to such automorphisms A notice that from A we obtain a permutation π_v of $\{0, ..., d-1\}$ at each vertex giving the rearrangement of its d immediate extensions. An automorphism of $\mathcal{T}_n$ will be represented by an assignment of a permutation of $\{0, ..., d-1\}$ to each vertex of the tree except for those in $b(\mathcal{T}_n)$. We attach and concatenate tree automorphisms as above.

Now we discuss the relevance of tree names and tree automorphisms to uniformly d to 1 endomorphisms. Let $(X, \sigma, \mu, \mathcal{F})$ be a uniformly d to 1 endomorphism and P be a partition of X. This implies that for almost every $x \in X$ the set $\sigma^{-1}(x)$ consists of d points. Also for any $y \in \sigma^{-1}(x)$

$$\mu(y \mid \sigma^{-1}(x)) = 1/d.$$

For such an endomorphism there exists a measurable d set partition K of X such that almost every x has one preimage in each element of K. Label the sets of K as K_0 through K_{d-1}. We now define a set of **partial inverses** for σ. Define for each $i \in \{0, ..., d-1\}$ and $x \in X$ define $\sigma_i^{-1}(x)$ to be the preimage of x in K_i. For $v = (v_1, \ldots, v_i) \in \mathcal{T}$ define $\sigma^v(x) = \sigma_{v_i}^{-1}(...(\sigma_{v_1}^{-1}(x)))$. Also define the tree name of x by $\mathcal{T}_x(v) = P(\sigma^v(x))$.

We now put a family of metrics on $\mathcal{T}, P$ names (and on $\mathcal{T}_n, P$ names). For two $\mathcal{T}, R$-names h and h' we define

$$\bar{t}_n^d(h, h') = \min_{A \in \mathcal{A}_n^d} \frac{1}{n} \sum_{0 < |v| \le n} d^{-|v|} \chi_{h(v) \neq h'(A(v))}.$$

Definition 2.1. *Let (X, σ, μ) be a uniform d to one endomorphism and P a tree adapted partition of X. We say (X, σ, μ) and P are* **tree very weak Bernoulli (tree v.w.B)** *if for any $\varepsilon > 0$ there exists an N and a set G with $\mu(G) > 1 - \varepsilon$ such that for any $x, y \in G$ we have*

$$\bar{t}_N^d(\mathcal{T}_x^d, \mathcal{T}_y^d) < \varepsilon.$$

The main result of [2] is the following.

Theorem 2.2. *A uniform d to one endomorphism (X, σ, μ) is isomorphic to the one sided Bernoulli d shift iff there exists a generating partition P so that (X, σ, μ) and P are tree v.w.B.*

In [2] the following variations are also proven.

Lemma 2.3. *The definition of tree very week Bernoulli is equivalent if the phrase "there exists an N and" is replaced by "and for all N sufficiently large there exists".*

Theorem 2.4. *If there exists a generating partition P so that (X, σ, μ) and P are tree v.w.B. then for any P', a generating partition, (X, σ, μ) and P' are tree v.w.B.*

In this paper we will be dealing with a uniformly 2-1 endomorphism (X, σ, μ). Its square is a uniformly 4-1 endomrophism. There is a strong relation between the tree name generated by x and σ and the tree name generated by x and σ^2. We now discuss that relationship. There is a natural (but not canonical) identification of $\mathcal{T}^2_{2k}$ with $\mathcal{T}^4_k$. Define $d_k(i)$ to be the kth digit in the dyadic expansion of i. The identification is done by taking $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_k) \in \mathcal{T}^4_k$ and associating it with

$$J(\bar{v}) = (d_2(\bar{v}_1), d_1(\bar{v}_1), \ldots, d_2(\bar{v}_k), d_1(\bar{v}_k)) \in \mathcal{T}^2_{2k}.$$

If (X, σ, μ) is a uniformly two to one endomorphism then (X, σ^2, μ) is a uniformly four to one endomorphism. For each $x \in X$ we form the $\mathcal{T}^4, P \times P$ name $\mathcal{T}^4_x$ such that for each $\bar{v} \in \mathcal{T}^4$

$$\mathcal{T}^4_x(\bar{v}) = (P(\sigma^{J(\bar{v})}(x)), P(\sigma^{J(\bar{v})|_{|\bar{v}|-1}}(x)))$$

where $\bar{v}|_k = \bar{v}_1\bar{v}_2 \ldots \bar{v}_k$. In a similar manner we can take any $\mathcal{T}^2, P$ name h and form a $\mathcal{T}^4, P \times P$ name $\bar{h}$.

We can also take a map $A \in \mathcal{A}^2_{2k}$ and form a map $\bar{A} \in \mathcal{A}^4_k$. But we can not take a general map $\bar{A} \in \mathcal{A}^4_k$ and form a map $A \in \mathcal{A}^2_{2k}$. If it were possible then the construcuction in this paper would not be possible. The essential difference between $\mathcal{A}^4_k$ and $\mathcal{A}^2_{2k}$ is apparent in Lemmas 3.1 to 3.3.

We end this section with some more notation which will be used throughout the paper. Now we give a version of the $\bar{t}$ metric for finite tree names defined on some finite irregular subtrees of $\mathcal{T}$. Let h be a finite T, P name and h' be a T', P name where T and T' are both subtrees of $\mathcal{T}^d$. Assume both T and T' have the property that each vertex has either 0 or d extensions. Let A be a function from a subset of T to a subset of T' which preserves the tree structure. We also require that if v has extensions in T and $A(v)$ has extensions in T'

then $A(vv')$ is defined for all $|v'| = 1$. Let $Z(A) = \sum d^{-|v|}$ where the sum is taken over all v such that $A(v)$ is defined and $|v| > 0$. Then define

$$\bar{t}^d(h, h', A) = \frac{1}{Z(A)} \sum d^{-|v|} \chi_{h(v) \neq h'(A(v))}.$$

The sum is taken over all v such that $A(v)$ is defined and $|v| > 0$. This definition is consistent with the definitions above because if $T = T' = \mathcal{T}_n$ then $\bar{t}^d_n(h, h') = \min_A \bar{t}^d(h, h', A)$.

The reason that we introduced this last definition is the following equality. If h is a T, P name let $\sigma^v(h)$ be given by $\sigma^v(h)(v') = h(vv')$. Given a $\mathcal{T}, P$ name h, an automorphism A, set $V \subset T$, and $v \in V$ we define T_v and A_v as follows. Let T_v consist of all $\tilde{v} \in V$ such that the largest contraction of $\tilde{v}$ in V is v. For $\tilde{v} \in T_v$ define $A_v(\tilde{v})$ by

$$A(v)A_v(\tilde{v}) = A(v\tilde{v}).$$

Then if $\{0\} \in V$

$$\bar{t}^d(h, h', A) = \frac{1}{Z(A)} \sum_{v \in V} d^{-|v|} Z(A_v) \bar{t}^d(\sigma^v(h), \sigma^{A(v)}(h'), A_v). \tag{1}$$

3. Construction

The construction will be done by cutting and stacking. Traditionally cutting and stacking techniques have been used to construct invertible transformations. The procedure starts with a partition P. Then for each k there is a measure on $P^{[-k,k]}$. These measure converges in the weak * topology to a measure ν on $P^{\mathbb{Z}}$. Then the transformation is $(P^{\mathbb{Z}}, \sigma, \nu)$ where σ is the shift, i.e. $\sigma(x)_i = x_{i+1}$.

Our cutting and stacking procedure we will construct a sequence of measures $\mu_{n,k}$ on $P^{\mathcal{T}_k}$. These measures will converge to μ_k. These measures μ_k will converge in the weak * topology to a measure μ' on $P^{\mathcal{T}}$. This projects to a measure μ on $X = P^{\mathbb{N}}$. Then we form the endomorphism $(P^{\mathbb{N}}, \sigma, \mu)$. Doing the cutting and stacking with tree names instead of with sequences will guarantee that $(P^{\mathbb{N}}, \sigma, \mu)$ be a dyadic endomorphism.

The most important part of this construction is in line 2. The relevant consequences of this line are indicated by Lemmas 3.1 to 3.3. Let the partition P consist of all elements of the form (a, b, c, d), where $a, b \in \mathbb{N}$, and $c, d \in \{0, 1\}$. For $(a, b, c, d) \in P$, define

$$\pi_1(a, b, c, d) = a, \ldots, \pi_4(a, b, c, d) = d.$$

The construction is inductive. The first stage is n_0 which we pick by lines 4 and 5 below. We will define 2^{n_0} $\mathcal{T}_{100}, P$ names $B^4_{n_0,i}$. At each stage $n > n_0$ we will define four subtrees, T^1_n, T^2_n, T^3_n, and T^4_n, of $\mathcal{T}^2$. We will also define

(1) B^1_n, a T^1_n, P name,

(2) $B^2_{n,i}$, $0 \le i \le 2^n - 1$, which are T^2_n, P names,
(3) $B^3_{n,i}$, $0 \le i \le 2^n - 1$, which are T^3_n, P names and
(4) $B^4_{n,i}$, $0 \le i \le 2^n - 1$, which are T^4_n, P names.

The names B^1_n, $B^2_{n,i}$, and $B^3_{n,i}$ are all auxiliary names. It is the names $B^4_{n,i}$ which will be used in the cutting and stacking procedure to define the measure.

In order to define these trees and tree names we will need some sequences. We will use sequences of integers $F(n), H(n)$ and $N(n)$. Also for each n and i, $0 \le i < 2^n$, we have a collection of "psuedorandom" sequences $S_{n,i}(j) \in \{0, ..., 2^n - 1\}$ for all $j \in \{1, ..., N(n)\}$.

Set $T^4_{n_0} = \mathcal{T}_{100}$. First we define $B^4_{n_0,i}$, which is a $\mathcal{T}_{100}, P$ name. For each $v \in \mathcal{T}_{100}$ we set

$$B^4_{n_0,i}(v) = (i, n_0, v_{|v|}, 0).$$

Now assume that $B^4_{n-1,i}$ has been defined for all $0 \le i < 2^{n-1}$. From this we will define $F(n), H(n), N(n)$ and $S_{n,i}(j)$ for all i and j, $0 \le i < 2^n$ and $1 \le j \le N(n)$. As the choice of these sequences is technical we delay this until Section 4. These sequences will allow us to define B^1_n, $B^2_{n,i}$, $B^3_{n,i}$, and $B^4_{n,i}$.

First we assume that $F(n)$, $N(n)$ and $S_{n,i}(j)$ have all been defined. We will use $F(n)$ to define T^1_n and B^1_n as follows. Let T^1_n be all vertices that satisfy $|v| \le 2.5F(n)$,

$$\sum_1^{|v|} v_i \le F(n) \text{ and } \sum_1^{|v|-1} v_i < F(n).$$

For $v \in T^1_n$ set

$$B^1_n(v) = (1, n, v_{|v|}, 0).$$

Define $h_{1,n}$ and $h_{2,n}$ to be $\mathcal{T}^2_2, P$ names as follows. Let

$$\begin{aligned} h_{1,n}(0) &= (2, n, 0, 0) & h_{2,n}(0) &= (2, n, 0, 0) \\ h_{1,n}(00) &= (2, n, 0, 0) & h_{2,n}(00) &= (2, n, 0, 0) \\ h_{1,n}(01) &= (2, n, 1, 0) & h_{2,n}(01) &= (2, n, 1, 1) \\ h_{1,n}(1) &= (2, n, 1, 0) & h_{2,n}(1) &= (2, n, 1, 0) \\ h_{1,n}(10) &= (2, n, 0, 1) & h_{2,n}(10) &= (2, n, 0, 0) \\ h_{1,n}(11) &= (2, n, 1, 1) & h_{2,n}(11) &= (2, n, 1, 1) \end{aligned}$$

Define $T^2_n = \mathcal{T}_{2n}$ and

$$B^2_{n,i} = h_{d_1(i),n} \circ h_{d_2(i),n} \circ \cdots \circ h_{d_n(i),n}.$$

Set $T_n^3 = T_n^2 \circ T_{n-1}^4$. The crucial aspect of the construction is how we attach T_{n-1}^4, P names to the bottom of $B_{n,i}^2$. This will be done in a different manner for each i. In Lemmas 3.1 to 3.3 we indicate why we use this method. Define

$$N_{n,i,k}(v) = \sum_{m=1}^{k} 2^{m-1}\pi_4(B_{n,i}^2(v|_{2m})).$$

Then define

$$B_{n,i}^3(v) = \begin{cases} B_{n,i}^2(v) & |v| \le 2n \\ B_{n-1,N_{n,i,n}(v)}^4(v_{2n+1}...v_{|v|}) & |v| > 2n. \end{cases} \tag{2}$$

Given that $B_{n,i}^3$ and $S_{n,i}(j)$ have been defined for all i and j let

$$B_{n,i}^4 = B_n^1 \circ B_{n,S_{n,i}(1)}^3 \circ B_{n,S_{n,i}(2)}^3 \circ \cdots \circ B_{n,S_{n,i}(N(n))}^3.$$

This completes the definition of $B_{n,i}^4$.

For each n and $k < n$ define

$$\mu_{n,k} = \frac{1}{2^n \sum_{v \in T_n^4} 2^{-|v|}} \sum_i \sum_{v \in T_n^4} 2^{-|v|}\delta_{m_{n,v,i,k}}$$

where $\delta_{m_{n,v,i,k}}$ is the point mass on the $\mathcal{T}_k^2$ name $\sigma^v(B_{n,i}^4)$. By the cutting and stacking procedure outlined at the beginning of the section the tree names $B_{n,i}^4$ for all n and i define a measure μ on $X = P^{\mathbb{N}}$. It is not difficult to show that (X, σ, μ) is a uniformly 2-1 endomorphism. We refrain from doing that here as it is an immediate consequence of Theorem 5.4

We now highlight two important aspects of the construction.

Lemma 3.1. *For any n, $k \le n$, and all $i, j \in \{0, ..., 2^{n-1} - 1\}$ there exists $\bar{A} \in \mathcal{A}_k^4$ such that for all $\bar{v} \in \mathcal{T}_k^4$ with $|\bar{v}| = k$*

$$N_{n,i,k}(J(\bar{v})) = N_{n,j,k}(J(\bar{A}(\bar{v})))).$$

Proof. There exists $\bar{A} \in \mathcal{A}_1^4$ such that for all $v \in \mathcal{T}_1^4$

$$N_{1,i,k}(J(\bar{v})) = N_{1,j,k}(J(\bar{A}(\bar{v}))).$$

The lemma follows easily by induction. □

This fact gives us the following lemma, which is the primary lemma that will be used in Section 5.

Lemma 3.2. *For any $n, i,$ and j*

$$\bar{t}^4(\bar{B}_{n,i}^3, \bar{B}_{n,j}^3) < 2n/H(n).$$

Proof. By Lemma 3.1 there exists a automorphism $\bar{A} \in \mathcal{A}_n^4$ such that for all $\bar{v}$ with $|\bar{v}| = n$

$$N_{n,i,n}(J(\bar{v})) = N_{n,j,n}(J(\bar{A}(\bar{v}))). \tag{3}$$

Then form the automorphism $\bar{A}'$ by attaching the identity automorphism onto every vertex with $|\bar{v}| = n$. By lines 2 and 3 we have that

$$\bar{B}^3_{n,i}(\bar{v}') = \bar{B}^3_{n,i}(\bar{A}'(\bar{v}'))$$

for all $|\bar{v}'| > n$. □

In contrast the following lemma shows that the above strategy won't work for σ. Define

$$D_k(i,j) = |\{l : \ 1 \le l \le k \text{ and } d_l(i) \ne d_l(j)\}|.$$

Lemma 3.3. *For any n, $k \le n$, $A \in \mathcal{A}^2_{2k}$ and any $i, j \in \{0, ..., 2^{n-1} - 1\}$*

$$|\{v : \ |v| = 2k \text{ and } N_{n,i,k}(v) = N_{n,j,k}(A(v))\}| \le 2^{2k - D_k(i,j)}.$$

Proof. Notice that for all $A \in \mathcal{A}^2_{2k}$ there are exactly two v with $|v| = 2$ and

$$\pi_4(h_1(v)) = \pi_4(h_2(A(v))).$$

The lemma follows easily by induction from this statement. □

The lemma above will form the basis of the proof that σ is not tree very week Bernoulli. We will show that for two arbitrary points x and y that

$$t_n(\mathcal{T}_x, \mathcal{T}_y) \not\to 0$$

as n approaches ∞. This will require a technical argument in Section 6.

Before we define the sequences some more notation is needed. For $v \in T^4_n$ we say that v is in top of a T^3_n block if

$$v \in b(T^1_n) \cup b(T^1_n \circ T^3_n) \cup \cdots \cup b(T^1_n \circ \underbrace{T^3_n \circ \cdots \circ T^3_n}_{N(n)-1}).$$

For $v' \in T^3_n$ we say that v' is in top of a T^4_{n-1} block if $|v'| = 2n$. Inductively we can go back and define what it means for $v \in T^4_n$ to be in the top of a T^4_k block or in the top (or bottom) of a T^3_k block for any $k \le n$. For a point $x \in X$ we say x is in a T^4_n block if $\pi_2(P(x)) \le n$. For such an x let $f(x)$ be the smallest nonnegative integer such that

$$y = \sigma^{f(x)}(x)$$

is not in a T^4_n block. Let $v_n(x)$ be such that $x = \sigma^{v_n(x)}(y)$. There exists a unique $j \le 2^n - 1$ such that $\mathcal{T}_y(v) = B^4_{n,j}(v)$ for all $v \in T^4_n$. Then set $\text{block}_n(x) = j$. We say x is in the top of a T^3_{n-1} block if $v_n(x)$ is in the top of a T^3_{n-1} block.

4. Choosing the sequences

The selection of the sequences in this section is similar to the selection of the "psuedorandom" sequences in previous cutting and stacking arguments such as those in [8]. Set $H(n_0) = 100$. We choose $F(n)$ so that $F(n) > H(n-1)$ and $F(n)$ is even.

Lemma 4.1. *There exists $c < 1$ such that for all n and all $i \leq 2^n - 1$*

$$|\{j :\ 0 \leq j < 2^n \text{ and } D_n(i,j) \leq 4n/10\}| \leq \sum_{k \leq 4n/10} \binom{n}{k} \leq 2^{cn}. \tag{4}$$

Proof. This is a consequence of the exponential convergence to the law of large numbers. □

The following lemma tells us describes the properties that we want $N(n)$ and $S_{n,i}(j)$ to have.

Lemma 4.2. *For all $n > n_0$ there exists $N(n)$ and sequences $S_{n,i} \in \{0, ..., 2^n - 1\}^{N(n)}$ for all $0 \leq i \leq 2^n - 1$ with the following property. For no i and j does there exist*

$$M : \{1, \dots, N(n)\} \to \{1, \dots, N(n)\},$$

and $W \subset \{1, \dots, N(n)\}$ such that

1. *$|W| \geq N(n)/200n^4$,*
2. *$M|_W$ is increasing, and*
3. *for all $k \in W$*

$$D_{n-1}(S_{n,i}(k), S_{n,j}(M(k))) < n/6.$$

Proof. If there exists i, j and W satisfying the above conditions then there exists satisfying the above conditions with $|W| = \lfloor N(n)/200n^4 \rfloor$, where $\lfloor x \rfloor$ indicates the greatest integer less than or equal to x. Given an element $a = (a(1), ..., a(N)) \in \{0, ..., 2^{n-1} - 1\}^N$ we will calculate the number of other such elements b for which there exist M and W such that for all $k \in W$

$$D_{n-1}(a(k), b(M(k))) < n/6.$$

We will give a bound for this number which depends only on N. We show that as $N \to \infty$ this bound divided by $2^{(n-1)N}$ goes to zero. Thus we can pick $N(n)$ so large that the bound divided by $2^{(n-1)N(n)}$ is less than $1/2^n$ which will let us pick the desired sequences.

Fix a. We will first count the number of b for which there exist M and W which satisfy the above conditions. The number of sets $W \subset \{1, \dots, N\}$ with $|W| = \lfloor N/200n^4 \rfloor$ is $\binom{N}{\lfloor N/200n^4 \rfloor}$ which is less than or equal to $(800n^4)^{N/200n^4}$.

Given a, W and $M(W)$ there are at most

$$\begin{aligned}(2^{n-1})^{N-|W|}(2^{c(n-1)})^{|W|} &= (2^{n-1})^N(2^{c(n-1)}/2^{n-1})^{|W|}\\ &\leq (2^{n-1})^N(2^{(c-1)(n-1)})^{|W|}\\ &\leq (2^{n-1})^N(2^{(c-1)(n-1)})^{N/200n^4}\end{aligned}$$

possible choices for b so that for all $k \in W$

$$D_{n-1}(a(k), b(M(k))) \leq 4n.$$

There are at most $(800n^4)^{N/200n^4}$ choices of W and $M(W)$. For each a, M, and W there are at most $(2^{n-1})^N(2^{(c-1)(n-1)})^{N/200n^4}$ choices of b. Thus for a given a there are at most

$$\left((800n^2)^{N/200n^4}\right)^2(2^{n-1})^N(2^{(c-1)(n-1)})^{N/200n^4} \leq (2^{n-1})^N\left(\frac{10^6n^4}{2^{(1-c)(n-1)}}\right)^{N/200n^4}$$

choices of b. We choose n_0 such that for $n > n_o$

$$\frac{10^6n^4}{2^{(1-c)(n-1)}} < 1. \tag{5}$$

Thus we can choose $N(n)$ so large that

$$\left(\frac{10^6n^4}{2^{(1-c)(n-1)}}\right)^{N(n)/200n^4} < 1/2^n. \tag{6}$$

We choose $S_{n,i}$ inductively. Choose $S_{n,0} \in \{0, ..., 2^{n-1}-1\}^{N(n)}$ in an arbitrary manner. Assume that $S_{n,0}, \ldots, S_{n,i}$ have been chosen. Line 6 ensures that the number of possible choices for $S_{n,i+1}$ is at least

$$(2^{n-1})^{N(n)} - (2^{n-1})^{N(n)}(i+1)\left(\frac{10^6n^4}{2^{(1-c)(n-1)}}\right)^{N(n)/200n^4} > 0.$$

Thus it is possible to choose $S_{n,0}$ up to $S_{n,2^n-1}$. □

We have one more condition to impose on $N(n)$. After we have chosen $N(n)$ we will set

$$H(n) = F(n) + N(n)(H(n-1) + 2n).$$

We also want $N(n)$ large enough so that

$$\left(1 - 2^{-10H(n-1)}\right)^{H(n)/5n^2H(n-1)} < 1/n. \tag{7}$$

Now choose $N(n)$ and $S_{n,i}$ so that they satisfy the hypothesis of Lemma 4.2.

This definition of $H(n)$ implies that for each n and $v \in b(T_n^4)$

$$H(n) \leq |v| < 2.5H(n). \tag{8}$$

There is some $v \in b(T_n^4)$ such that the lower bound is achieved. Also $H(n)$ is even.

5. (X, σ^2, μ) is tree very week Bernoulli

Remember that for any $T \subset \mathcal{T}^2$ we have defined $\bar{T}$ and

$$\bar{T} = \{\bar{v} \in \mathcal{T}^4 : \; J(\bar{v}) \in T\}.$$

For any B, a T, P name, we have defined $\bar{B}$ an $\bar{T}, P \times P$ name by

$$\bar{B}(\bar{v}) = (B(J(\bar{v}), B(J(\bar{v})|_{2|\bar{v}|-1})))$$

for every $\bar{v} \in \bar{T}$.

We also define

$$R_n(v) = \min |v'|$$

for any $v \in T_n^4$, where the minimum is over all v' such that $vv' \in b(T_n^4)$. Note that by line 8

$$\max R_n(v) = H(n).$$

Finally set

$$L_n(v) = \begin{cases} 0 & \text{if } v \in T_n^1 \\ N(n) - \lfloor R_n(v)/(H(n-1)+2n) \rfloor & \text{otherwise.} \end{cases}$$

Thus $L_n(v) = i$ if v is in the ith T_n^3 tree in the T_n^4 tree.

Lemma 5.1. *For any n, i, and j there exists $\bar{A} : \bar{T}_n^4 \to \bar{T}_n^4$*

$$\bar{t}^4(\bar{B}_{n,i}^4, \bar{B}_{n,j}^4, \bar{A}) < 1/n.$$

Proof. The map $\bar{A}$ will be constructed in such a way that

$$\bar{v}|_{|\bar{v}|} = \bar{A}(\bar{v})|_{|\bar{v}|}$$

unless

$$\pi_1 \times \pi_1(\bar{B}_{n,i}^4(\bar{v})) = (2, 2).$$

This will ensure that for every $\bar{v}$

$$R_n(J(\bar{v})) = R_n(J(\bar{A}(\bar{v})))$$

which implies that $J(\bar{v})$ is in the top of a j block if and only if $J(\bar{A}(\bar{v}))$ is in the top of a j block.

The map $\bar{A}$ is defined inductively. Pick a $\bar{v}$ such that $\bar{A}(\bar{v})$ has not been defined yet and $|\bar{v}|$ is minimal with this property. Define

$S(\bar{v}) = \{\bar{v}' \;\; :$

(1) $\bar{v}' = \bar{v}$ or $\bar{v}'$ is an extension of $\bar{v}$
(2) $J(\bar{v}')$ is in the top of a T_k^3 block with $\frac{3}{4}n \leq k \leq n$ and
(3) no contraction of $\bar{v}'$ satisfies 1 and 2.}

If $\bar{v}' \in S(\bar{v})$ then set

$$\bar{A}(\bar{v}')|_{|\bar{v}'|} = \bar{v}'|_{|\bar{v}'|}.$$

For each $\bar{v}' \in S(\bar{v})$ Lemma 3.2 provides an automorphism $\bar{A}_{\bar{v}'}$. Extend $\bar{A}$ by attaching $\bar{A}_{\bar{v}'}$ onto $\bar{v}'$. This inductively defines $\bar{A}$.

Let G consist of all $\bar{v} \in \bar{T}_n^4$ such that $J(\bar{v})$ is in a $\lfloor 3n/4 \rfloor$ block. For any $\bar{v} \in G$ such that there exists some $k > 3n/4$ and $\bar{v}'$ such that $J(\bar{v}')$ is at the top the k block that contains $J(\bar{v})$ then

$$\bar{B}_{n,i}^4(\bar{v}) = \bar{B}_{n,j}^4(\bar{A}(\bar{v})).$$

The fraction of $\bar{v} \in G$ which do not satisfy the last condition is equal to $2^{-n/4}$. Thus

$$\begin{aligned} t^4(\bar{B}_{n,i}^4, \bar{B}_{n,j}^4, \bar{A}) &< \left(1 - \frac{\sum_{\bar{v} \in G} 4^{-|\bar{v}|}}{\sum_{\bar{v} \in \bar{T}_n^4} 4^{-|\bar{v}|}}\right) + 2^{-n/4} \\ &< 1/n. \end{aligned}$$

□

Corollary 5.2. *Let h and h' be any $\mathcal{T}_1^2, P$ names. Let H and H' be $\mathcal{T}_1^2 \circ T_n^4, P$ names defined by*

$$H = h \circ B_{i,n}^4 \text{ and } H' = h' \circ B_{j,n}^4.$$

Then there exists $\bar{A}$ such that

$$\bar{t}^4(\bar{H}, \bar{H}', \bar{A}) < 1/n.$$

Proof. The proof is the same as the previous lemma. □

Lemma 5.3. *Given n and $v, v' \in T_n^4$ so that $R_n(v), R_n(v') \geq H(n)/n$ there exists $A \in \mathcal{A}_{2\lfloor H(n)/n^2 \rfloor}^2$ such that*

$$\frac{1}{2^{2\lfloor H(n)/n^2 \rfloor}} |\{\tilde{v} : \ |\tilde{v}| = 2\lfloor H(n)/n^2 \rfloor \text{ and } R_n(v\tilde{v}) \neq R_n(v'A(\tilde{v})) \bmod H(n-1)\}| \leq 1/n.$$

Proof. We will define A inductively. By the choice of $F(n)$ there exists $\tilde{v}$ and $\tilde{v}'$ such that $|\tilde{v}| = |\tilde{v}'| < 10H(n-1)$ and

$$R_n(v\tilde{v}) = R_n(v'\tilde{v}') \bmod H(n-1). \tag{9}$$

Define A for all $|\hat{v}| \leq |\tilde{v}|$ in any way such that $A(\tilde{v}) = \tilde{v}'$. If $A(\hat{v})$ has been defined and

$$R_n(v\hat{v}) = R_n(v'A(\hat{v})) \bmod H(n-1)$$

then attach the identity on $\hat{v}$ (i.e. for all $v'' \in \mathcal{T}$ let $A(\hat{v}v'') = A(\hat{v})v''$). If $A(\hat{v})$ has been defined and

$$R_n(v\hat{v}) \neq R_n(v'A(\hat{v})) \bmod H(n-1)$$

then define A for all such that vertices in $v \circ 10H(n-1)$ for one $\tilde{v} \in v \circ 10H(n-1)$

$$R_n(v\hat{v}\tilde{v}) = R_n(v'A(\hat{v}\tilde{v})) \bmod H(n-1).$$

This is possible by line 9. Continue in this manner until A is defined for $\mathcal{T}_{2\lfloor H(n)/n^2 \rfloor}$. Then

$$\frac{1}{2^{10kH(n-1)}}|\{\tilde{v} : |\tilde{v}| = 10kH(n-1) \text{ and } R_n(v\tilde{v}) = R_n(v'A(\tilde{v})) \bmod H(n-1)\}| \leq (1-2^{-10H(n-1)})^k.$$

So if $k = H(n)/5n^2H(n-1)$ then

$$\frac{1}{2^{2\lfloor H(n)/n^2 \rfloor}}|\{\tilde{v} : |\tilde{v}| = 2\lfloor H(n)/n^2 \rfloor \text{ and } R_n(v\tilde{v}) = R_n(v'A(\tilde{v})) \bmod H(n-1)\}| \leq 1/n$$

because of line 7. □

Theorem 5.4. *(X, σ^2, μ) is isomorphic to the one sided Bernoulli 4 shift.*

Proof. Let

$$G_n = \{x : x \text{ is in a } T_n^4 \text{ block, } R_n(v_n(x)) > H(n)/n, \text{ and } L_n(v_n(x)) \geq 1\}.$$

We have that

$$\mu(G_n) \geq \mu\{x : x \text{ is in a } T_n^4 \text{ block}\} - 1/n - 3F(n)/H(n)$$

and

$$\lim_n \mu(G_n) = 1. \tag{10}$$

Given $x, y \in G_n$ we get $v_n(x), v_n(y), i, j$ such that

$$\mathcal{T}_x(\tilde{v}) = \sigma^{v_n(x)}(B_{n,i}^4(\tilde{v}))$$

for all $\tilde{v} \in \mathcal{T}_{\lfloor H(n)/n \rfloor}$ and

$$\mathcal{T}_y(\tilde{v}) = \sigma^{v_n(y)}(B_{n,j}^4(\tilde{v}))$$

for all $\tilde{v} \in \mathcal{T}_{\lfloor H(n)/n \rfloor}$. Then by Lemma 5.3 there exists $A \in \mathcal{A}^2_{2\lfloor H(n)/n^2 \rfloor}$. This generates $\bar{A} \in \mathcal{A}^4_{\lfloor H(n)/n^2 \rfloor}$. Extend $\bar{A}$ by the identity until the bottom (or next to bottom) of $n-1$ trees. Then attach automorphisms of the form in lemma 5.1 and corollary 5.2 to form $\bar{A}'$. This automorphism $\bar{A}'$ has the property that if $|\bar{w}| > H(n)/n^2 + H(n-1)$ and $vJ(\bar{w})$ is not in the top n levels of a T_{n-1}^3 block then

$$P(\sigma^{vJ(\bar{w})}(B_{n,i}^4)) = P(\sigma^{v'J(\bar{A}'(\bar{w}))}(B_{n,j}^4))$$

and

$$P(\sigma^{v(J(\bar{w})|_{2|\bar{w}|-1})}(B^4_{n,i})) = P(\sigma^{v'(J(\bar{A}'(\bar{w}))|_{2|\bar{w}|-1})}(B^4_{n,j})).$$

Thus

$$\begin{aligned}
\bar{t}^4_{\lfloor H(n)/2n\rfloor}(\bar{\mathcal{T}}^4_x, \bar{\mathcal{T}}^4_y) &= \bar{t}^4_{\lfloor H(n)/2n\rfloor}(\overline{\sigma^{v_n(x)}(B^4_{i,n})}, \overline{\sigma^{v_n(y)}(B^4_{j,n})}) \\
&\leq \bar{t}^4_{\lfloor H(n)/2n\rfloor}(\overline{\sigma^{v_n(x)}(B^4_{i,n})}, \overline{\sigma^{v_n(y)}(B^4_{j,n})}, \bar{A}') \\
&\leq \frac{H(n)/n^2 + H(n-1)}{H(n)/2n} + \frac{2n}{H(n-1)} \\
&\leq \frac{2}{n} + \frac{2nH(n-1)}{H(n)} + \frac{2n}{H(n-1)}.
\end{aligned}$$

This last expression goes to zero as n goes to ∞. Combined with line 10 this shows that (X, σ^2, μ) and $P \times P$ are tree very week Bernoulli. As $P \times P$ is tree adapted and generating Theorem 2.2 implies that (X, σ^2, μ) is isomorphic to the one sided Bernoulli 4 shift. □

6. (X, σ, μ) is not tree very week Bernoulli

In this section we show that (X, σ, μ) and P are not tree very week Bernoulli. That is we will show that there exists an $\epsilon > 0$ such that for most x and y

$$\bar{t}^2_n(\mathcal{T}_x, \mathcal{T}_y) > \epsilon$$

for arbitrarily large n. This proves that (X, σ, μ) is not isomorphic to the one sided Bernoulli two shift.

Define the sequence ϵ_n by $\epsilon_{n_0} = 1$ and

$$\epsilon_n = \epsilon_{n-1}(1 - 5/(n-1)^2)(1 - 2^{-.4n}).$$

Then $\epsilon = \lim_n \epsilon_n > 0$. The fundamental lemma of this section is as follows.

Lemma 6.1. *Given any n, i, j, and $v \in T^4_n$ such that $i \neq j$ and $R_n(v) > H(n)/n^2$*

$$\bar{t}^2(\sigma^v(B^4_{n,i}), B^4_{n,j}) \geq \epsilon_n.$$

First we sketch the proof which is done by induction. Given n and v define

$$S_1(n, v) = \{\tilde{v} : L_n(v\tilde{v}) \geq 1 \text{ and } R_n(v\tilde{v}|_{|v\tilde{v}|-1}) = 0 \bmod H(n-1) + 2n\} \cup \{0\}.$$

Thus $S_1 \setminus \{0\}$ is the set of vertices $\tilde{v}$ such that $v\tilde{v}$ is in the top of a T^3_n block. Also define

$$S_2(n, A) = \{\tilde{v} : L_n(A(\tilde{v})) \geq 1 \text{ and } R_n(A(\tilde{v})|_{|\tilde{v}|-1}) = 0 \bmod H(n-1) + 2n)\}$$

which is the set of vertices such that $A(\tilde{v})$ is in the top of a T^3_n block.

Given n, v and A every vertex $\tilde{v}$ such that $v\tilde{v} \in b(T_n^4)$ generates a map $M_{\tilde{v}}$ in the following way. To define $M_{\tilde{v}}(i)$ find the vertex v^* such that $vv^* \in S_1$ such that v^* is a contraction of $\tilde{v}$ and $L_n(vv^*) = i$ if it exists. Define

$$M_{\tilde{v}}(i) = L_n(A(v^*)).$$

$M_{\tilde{v}}$ is always nondecreasing. By line 8 for any $\tilde{v}$ and k

$$M_{\tilde{v}}(k+3) > M_{\tilde{v}}(k). \tag{11}$$

Using S_2 we can define $N_{A^{-1}(\tilde{v})}$ in a corresponding manner. Then we show that if the induction hypothesis is not satisfied then there exists a $\tilde{v}$ such that $M_{\tilde{v}}$ or $N_{A^{-1}(\tilde{v})}$ does not satisfy Lemma 4.2.

For the rest of the section fix v, i, j, n and A such that $R_n(v) > H(n)/n^2$. For $v^* \in S_1 \cup S_2$ define T_{v^*} to consist of all v' such that $v^*v' \in T_n^4$ and for no contraction v'' of v' is $v^*v'' \in S_1 \cup S_2$. Let $B_{A(v^*)}(v') = B_{n,j}^4(A(v^*v'))$. Let $A_{v^*}(v')$ be defined by

$$A(v^*)A_{v^*}(v') = A(v^*v').$$

We say that $v^* \in S_1 \cup S_2$ is **good** if

(1) $Z(A_{v^*}) \geq 2H(n-1)/(n-1)^2$ and
(2) $\bar{t}(\sigma^{vv^*}(B_{n,i}^4), \sigma^{A(v^*)}(B_{n,j}^4), A_{v^*}) \leq \epsilon_{n-1}(1 - 2^{-.4n})$.

Lemma 6.2. *If Lemma 6.1 is true for $n-1$ and v^* is good then*

$$D_{n-1}(S_{n,i}(L_n(vv^*)), S_{n,j}(L_n(A(v^*)))) \leq .4n.$$

Proof. Since v^* is good $Z(A_{v^*}) \geq 2H(n-1)/(n-1)^2$ and we have that

$$R_{n-1}(v^*) \geq H(n-1)/(n-1)^2.$$

Thus the second condition of Lemma 6.1 holds for all portions of $n-1$ blocks inside T_{v^*}. By Lemma 3.3 the first condition of Lemma 6.1 applies to a fraction at least

$$1 - 2^{-D_{n-1}(S_{n,i}(L_n(vv^*)), S_{n,j}(L_n(v'A(v^*))))}$$

of the B_{n-1}^3 trees directly below v^*. If this number is less than $1 - 2^{-.4n}$ then

$$D_{n-1}(S_{n,i}(L_n(vv^*)), S_{n,j}(L_n(v'A(v^*)))) < .4n$$

□

Lemma 6.3. *Assume that Lemma 6.1 is true for $n-1$ and*

$$\vec{t}^2(\sigma^v(B^4_{n,i}), B^4_{n,j}, A) \leq \epsilon_n. \tag{12}$$

Then

$$\sum_{good\ v^*} 2^{-|v^*|} Z(A_{v^*}) > 2Z(A)/n^2. \tag{13}$$

Proof. By lines 1 and 12

$$\vec{t}^2(\sigma^v(B^4_{n,i}), B^4_{n,j}, A) = \frac{1}{Z(A)} \sum_{v^* \in S_1 \cup S_2} 2^{-|v^*|} Z(A_{v^*}) \vec{t}^2(\sigma^{vv^*}(B^4_{n,i}), \sigma^{A(v^*)}(B^4_{n,j}), A_{v^*}) \leq \epsilon_n \tag{14}$$

By the definition of $Z(A)$ and $Z(A_{v^*})$

$$\frac{1}{Z(A)} \sum_{v^* \in S_1 \cup S_2} 2^{-|v^*|} Z(A_{v^*}) = 1. \tag{15}$$

Thus combining lines 14 and 15

$$\frac{1}{Z(A)} \sum_{v^* \in S_1 \cup S_2, \vec{t}^2(\sigma^{vv^*}(B^4_{n,i}), \sigma^{A(v^*)}(B^4_{n,j}), A_{v^*}) \leq \epsilon_{n-1}(1-2^{-.4n})} 2^{-|v^*|} Z(A_{v^*}) \geq 5/(n-1)^2.$$

So by the definition of a vertex being good and the fact that

$$\frac{1}{Z(A)} \sum_{v^* \in S_1 \cup S_2, Z(A_{v*}) \leq 2H(n-1)/(n-1)^2} 2^{-|v^*|} Z(A_{v^*}) \leq 2/(n-1)^2$$

we have that

$$\sum_{\text{good } v^*} 2^{-|v^*|} Z(A_{v^*}) \geq 2Z(A)/n^2.$$

□

Lemma 6.4. *If $R_n(v) > H(n)/n^2$, $L_n(v) \geq 1$, Lemma 6.1 is true for $n-1$, and line 12 is satisfied then there exists $\tilde{v} \in b(T^4_n)$ and a set $W_{\tilde{v}} \subset \{1, \ldots, N(n)\}$ where the following conditions are satisfied:*

(1) *$M_{\tilde{v}}|_{W_{\tilde{v}}}$ is increasing*
(2) *$|W_{\tilde{v}}| \geq N(n)/200n^4$, and*
(3) *for all $k \in W_{\tilde{v}}$*

$$D_{n-1}(S_{n,i}(k), S_{n,j}(M_{\tilde{v}}(k))) < n/6$$

or

$3'$ *for all $k \in W_{\tilde{v}}$*

$$D_{n-1}(S_{n,i}(N_{A^{-1}(\tilde{v})}(k)), S_{n,j}(k)) < n/6.$$

Proof. If line 12 is satisfied then by Lemma 6.3

$$\sum_{\text{good } v^*} 2^{-|v^*|} Z(A_{v^*}) > 2Z(A)/n^2$$

By line 8 we have $Z(A_{v^*}) < 3H(n-1)$ and

$$3H(n-1) \sum_{\text{good } v^*} 2^{-|v^*|} > 2Z(A)/n^2$$

Thus

$$\sum_{\text{good } v^*} \left(\sum_{\tilde{v}} 2^{-|\tilde{v}|} \right) > 2Z(A)/3H(n-1)n^2$$

where the last sum is taken over all $\tilde{v}$ that are extensions of v^* and $v\tilde{v} \in b(T_n^3)$.

Given $\tilde{v}$ such that $v\tilde{v} \in b(T_n^3)$ define

$$W'_{\tilde{v}} = \{v'' : \ v'' \text{ is a contraction of } \tilde{v} \text{ and } vv'' \text{ is good}\}.$$

Thus by Fubini's theorem

$$\sum_{\tilde{v} \,:\, v\tilde{v} \in b(T_n^3)} 2^{-|\tilde{v}|} |W'_{\tilde{v}}| > 2Z(A)/3H(n-1)n^2.$$

By line 8 we have that

$$\sup_{\tilde{v}} |W'_{\tilde{v}}| \leq 3Z(A)/H(n-1).$$

Assign each $\tilde{v}$ such that $v\tilde{v} \in b(T_n^3)$ a weight of $2^{-|\tilde{v}|}$. Combined with Chebychev's inequality we have that the weighted fraction of $\tilde{v}$ such that $v\tilde{v} \in b(T_n^3)$ where

$$|W'_{\tilde{v}}| > Z(A)/20H(n-1)n^2$$

is at least $1/10n^2$. Since $R_n(v) > H(n)/n^2$ we have that $Z(A) > N(n)H(n-1)/n^2$. Thus there is a $\tilde{v}$ such that $v\tilde{v} \in b(T_n^3)$ and

$$\begin{aligned} |W_{\tilde{v}}| &\geq \frac{Z(A)}{20H(n-1)n^2} \\ &\geq \frac{N(n)H(n-1)/n^2}{20H(n-1)n^2} \\ &\geq \frac{N(n)}{20n^4}. \end{aligned}$$

Pick such a $\tilde{v}$. For any $\hat{v} \in W_{\tilde{v}} \cap S_1$

$$D_{n-1}(S_{n,i}(L_n(v\hat{v})), S_{n,j}(M_{\tilde{v}}(L_n(\hat{v})))) < n/6 \qquad (16)$$

by Lemma 6.2. For any $\hat{v} \in W_{\tilde{v}} \cap S_2$

$$D_{n-1}(S_{n,j}(L_n(\hat{v})), S_{n,i}(N_{A^{-1}(\tilde{v})}(L_n(v\hat{v})))) < n/6$$

by Lemma 6.2.

Either

$$|W'_{\tilde{v}} \cap S_1| > N(n)/40n^2 \text{ or } |W'_{\tilde{v}} \cap S_2| > N(n)/40n^2.$$

Without loss of generality it is the former. By line 11 we have that

$$M_{\tilde{v}}(k+3) > M_{\tilde{v}}(k).$$

Thus there exists $a \in \{0,1,2\}$ such that if we set

$$W_{\tilde{v}} = \{k : k = a \bmod 3 \text{ and } \exists \tilde{v} \text{ such that } \tilde{v} \in W'_{\tilde{v}} \cap S_1 \text{ with } L_n(v\tilde{v}) = k\}$$

then $|W_{\tilde{v}}| > N(n)/120n^4$, $M_{\tilde{v}}|_{W_{\tilde{v}}}$ is increasing and the third condition is satisfied by line 16. □

Proof of Lemma 6.1: The proof is by induction. The case $n = n_o$ is trivial. Assume that the lemma is true for $n-1$. Given $v, A, i,$ and j. By Lemma 4.2 $M_{\tilde{v}}$ does not satisfy the conclusion of lemma 6.4 for any $\tilde{v}$. Thus the hypothesis of Lemma 6.4 does not hold and

$$t^2(\sigma^v(B^4_{n,i}), B^4_{n,j}, A) > \epsilon_n$$

Thus Lemma 6.1 is true for n. □

We will also need a slightly different version of Lemma 6.1.

Lemma 6.5. *Given any $n, i \neq j$, and $v, v' \in T^4_n$ such that $R_n(v), R_n(v') > H(n)/2$*

$$\bar{t}^2_{H(n)/2}(\sigma^v(B^4_{n,i}), \sigma^{v'}(B^4_{n,j})) \geq \epsilon_n.$$

Proof. The proof is analogous to the inductive step in the proof of Lemma 6.1. Set

$$S_1(n,v) = \{\tilde{v} \ : |\tilde{v}| < |H(n)/2|, L_n(v\tilde{v}) \geq 1 \text{ and } \ R_n(v\tilde{v}|_{|v\tilde{v}|-1}) = 0 \bmod H(n-1) + 2n\}.$$

Also define

$$S_2(n,A,v') = \{\tilde{v} : |\tilde{v}| < |H(n)/2|, L_n(\tilde{v}) \geq 1 \text{ and } \ R_n(A(\tilde{v})|_{|\tilde{v}|-1}) = 0 \bmod H(n-1) + 2n)\}.$$

Then the proofs of Lemmas 6.3 and 6.4 go through by replacing $b(T^3_n)$ with $b(v \circ \mathcal{T}_{H(n)/2})$ and $A(*)$ with $v'A(*)$. □

Theorem 6.6. *(X, σ, μ) is not isomorphic to the one sided Bernoulli two shift.*

Proof. Let

$$Y_{1,n} = \{x \ :$$

(1) x is in an T_n^4 block,
(2) $0 \leq \text{block}_n(x) < 2^{n-1}$, and
(3) $R_n(v_n(x)) > H(n)/2\}$

and

$Y_{2,n} = \{x \ :$

(1) x is in an T_n^4 block,
(2) $\text{block}_n(x) \geq 2^{n-1}$, and
(3) $R_n(v_n(x)) > H(n)/2\}$

If $y_1 \in Y_{1,n}$ and $y_2 \in Y_{2,n}$ then by Lemma 6.5

$$\vec{t}^2_{H(n)/2}(\mathcal{T}_{y_1}, \mathcal{T}_{y_2}) > \epsilon_n > \epsilon > 0.$$

We also have that $\mu(Y_{1,n}), \mu(Y_{2,n}) > 1/10$. Thus (X, σ, μ) and P are not tree very weak Bernoulli. P is a tree adapted generating partition. Thus by Theorems 2.2 and 2.4 and Lemma 2.3 (X, σ, μ) is not isomorphic to the one sided Bernoulli two shift. □

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