A ZERO ENTROPY T SUCH THAT THE [T,ID] ENDOMORPHISM IS NONSTANDARD

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Abstract. We present an example of an ergodic transformation T, a variant of a zero entropy non loosely Bernoulli map of Feldman [1], such that the sequence of random variables generated by the [T,Id] endomorphism is nonstandard.

1. Introduction

Any sequence of random variables, $Y_0, Y_1, Y_2, ...$, defined on the space Y produces a decreasing sequence of σ-algebras $\mathcal{F}_n$. These are defined by $\mathcal{F}_n = \sigma(Y_n, Y_{n+1}, ...)$. The sequence Y_i is called **exact** if $\cap \mathcal{F}_n = \emptyset$. An **isomorphism** between two such sequences $\mathcal{F}_n$ and $\mathcal{G}_n$ is a 1-1, invertible, measure preserving map $\phi : \mathcal{F}_0 \to \mathcal{G}_0$ such that $\phi(\mathcal{F}_n) = \mathcal{G}_n$ $\forall n$. A sequence of random variables Y_i is called **standard** if there exists an independent sequence of random variables X_i producing an isomorphic sequence of σ-algebras. An equivalent definition is that there exists a sequence of independent partitions $\mathcal{I}_n$ such that $\mathcal{F}_n = \bigvee_{i=n}^{\infty} \mathcal{I}_i$. An endomorphism is said to be **standard** if the sequence of random variables produced by the endomorphism and a generating partition is standard.

Let T be any 1-1 map on $(Y, \mathcal{C}, \nu)$. Let S be the shift on $(X, \mathcal{B}, \mu) = (\{0,1\}^{\mathbb{N}}$, Borel σ-algebra, product measure $(\frac{1}{2}, \frac{1}{2}))$. Define [$T$,Id] on $(X \times Y, \mathcal{F}, \mu \times \nu)$ where $\mathcal{F} = \mathcal{B} \times \mathcal{C}$ by $[T,\mathrm{Id}](x, y) = (Sx, T^{x_0} y)$. [$T$,Id] is 2-1, since any point (x, y) has the preimages $(0x, y)$ and $(1x, T^{-1}y)$. Meilijson proved that [T,Id] is exact whenever T is ergodic [6].

This paper gives an example of a zero entropy T, which is a variant of a zero entropy non-loosely Bernoulli transformation of J. Feldman [1]. The corresponding [T,Id] endomorphism is nonstandard. The arguments in this paper can be easily be modified to show that for this T the $[T, T^{-1}]$ endomorphism is nonstandard.

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In [4] Heicklen and Hoffman proved that whenever T has positive entropy then the $[T, T^{-1}]$ endomorphism, which has an analogous definition to the [T,Id] endomorphism, is nonstandard. Feldman and Rudolph proved in [2] that whenever T is rank 1 then [T,Id] is standard. This argument can easily be modified to show that if T is rank 1 and T^2 is ergodic then the $[T, T^{-1}]$ endomorphism is standard. The present paper shows that T having positive entropy is not the only obstruction for $[T, T^{-1}]$ being standard.

In [5] Heicklen, Hoffman, and Rudolph showed a connection between between the $\bar{f}$ metric, which is involved in the definition of loosely Bernoulli, and the $\bar{v}$ metric, which is involved in the criteria for standardness. By using a non loosely Bernoulli transformation to create a nonstandard [T,Id] endomorphism, the present paper strengthens this connection.

2. Standardness Criteria

In this section we introduce the terminology necessary to state the standardness criteria. An **n branch** is an element of $\{0, 1\}^n$. An **n tree** is a binary tree of height n consisting of 2^n branches. The top level is b_0 and the bottom level is b_{n-1}. Let $\mathcal{A}_n$ be the set of automorphisms of an n tree.

For $m \leq n$ an **m tree inside an n tree** is a tree with 2^m branches such that the first $n-m$ coordinates all agree and the last m coordinates vary over all possibilities. A **labeled n tree** for a partition P over a point $y \in Y$ assigns to each branch b the label $P(T^{\sum b_i} y)$. The Hamming metric on labeled n trees is given by

$$d_n(W, W') = \frac{\#\text{ of branches where the labels of } W \text{ and } W' \text{ disagree}}{2^n}.$$

Fix P and let W and W' be labeled n trees over y and y' respectively. Define

$$v_n^P(y, y') = \inf_{a \in \mathcal{A}_n} d_n(aW, W').$$

In the case that $\mathcal{F}_n$ comes from [T,Id], Vershik's standardness criterion is the following.

Theorem 2.1. [Vershik] [7] *$\mathcal{F}_n$ is standard if and only if for every finite partition P, $\int v_n^P(y, y') d(\nu \times \nu) \to 0$.*

Remark 2.1. *A possibly more accessible proof of this can be found in* [3].

We also present some terminology used in the proof. A branch b in the tree over y **lands at a point z** if $z = T^{\sum b_i} y$. A subtree of height

k in the n tree over y **lies over a point z** if for any branch b in the subtree $z = T^{\sum_0^{n-k-1} b_i} y$.

3. Description of T

As we said in the introduction the transformation T will be a variant of one constructed by J. Feldman [1]. His example showed that there are ergodic transformations which are not loosely Bernoulli. The ergodic transformation presented below is also not loosely Bernoulli. The proof of this is the same as the proof for Feldman's transformation.

Our space is a subset of $\Omega = \{a_1, a_2, ..., a_{100}\}^{\mathbf{Z}}$ which is defined by a cutting and stacking procedure. The partition P is the partition into 100 sets depending on the symbol y_0. There exist 100 different 1 blocks, each of length 1, with $B_1^i = a_i$. For every n each n block has the same measure.

Now suppose we have $n-1$ blocks, $B_{n-1}^1, B_{n-1}^2, ..., B_{n-1}^{N(n-1)}$. We construct $N(n) = n^{10}$ different n blocks in the following way.

$$
\begin{aligned}
B_n^1 &= \left[\left(B_{n-1}^1\right)\left(B_{n-1}^2\right)...\left(B_{n-1}^{N(n-1)}\right)\right]^{(n^{10})^{(n^{10})}} \\
B_n^2 &= \left[\left(B_{n-1}^1\right)^{n^{10}}\left(B_{n-1}^2\right)^{n^{10}}...\left(B_{n-1}^{N(n-1)}\right)^{n^{10}}\right]^{(n^{10})^{(n^{10}-1)}} \\
B_n^i &= \left[\left(B_{n-1}^1\right)^{(n^{10})^{i-1}}\left(B_{n-1}^2\right)^{(n^{10})^{i-1}}...\left(B_{n-1}^{N(n-1)}\right)^{(n^{10})^{i-1}}\right]^{(n^{10})^{(n^{10}-i+1)}} \\
B_n^{N(n)} &= \left[\left(B_{n-1}^1\right)^{(n^{10})^{(n^{10}-1)}}\left(B_{n-1}^2\right)^{(n^{10})^{(n^{10}-1)}}...\left(B_{n-1}^{N(n-1)}\right)^{(n^{10})^{(n^{10}-1)}}\right]^{(n^{10})}
\end{aligned}
$$

Thus at stage n we have $N(n) = n^{10}$ different n blocks, each of length $l(n) = l(n-1)N(n-1)(n^{10})^{(n^{10})}$. T is the shift.

We will make one definition about the block structure of T which is important in the next section. An n **region** in n block B_n^i is a maximal subword in the n block which consists of one $n-1$ block repeated many times. The length of an n region in an n block B_n^i is $l(n-1)(n^{10})^{i-1}$.

4. $[T,\text{Id}]$ is Nonstandard

The proof is by induction. Excluding small sets the argument is as follows. We will define a sequence of heights h_n increasing to infinity and a sequence ϵ_n which converges down to $\epsilon > 0$. We will show that for y in n block B_j^i, y' in n block B_n^j with $j > i$ then $v_n(y, y') > \epsilon_n$. This statement will be enough to show that $[T,\text{Id}]$ is nonstandard. We show this by defining a height $k_n = k_n^{i,j}$ so that h_{n-1} subtrees of a k_n subtree of the h_n tree over y lie over many different $n-1$ blocks. But h_{n-1} subtrees of a k_n subtree over y' lie over only one $n-1$ block. A tree automorphism must take k_n subtrees to k_n subtrees, and h_{n-1} subtrees to h_{n-1} subtrees. By the inductive hypothesis the $v_{h_{n-1}}$ distance between most of the h_{n-1} subtrees paired by a tree automorphism is greater than ϵ_{n-1} so the v_{h_n} distance between the h_n trees is greater than ϵ_n.

In order to do the base case of the induction we need the following lemma.

Lemma 4.1. *For any $y = y_0, ..., y_{h-1}$, such that y is not of period 2, then the only other point y' such that $v_h(y, y') = 0$ is $y'_i = y_{h-1-i}$.*

Proof. A proof of this lemma is contained in [4]. □

At various times we will use the following fact about binomial distributions.

Lemma 4.2. *Given n and k, define $p_i = \sum_{j=i \bmod k} \binom{n}{j}$. Then*

$$\sum_{i=0}^{k-1} \left| \frac{p_i}{2^n} - \frac{1}{k} \right| \leq \frac{2k}{\sqrt{n}}.$$

Proof. The binomial n distribution and another copy of it translated by k differ by at most $\frac{k2^n}{\sqrt{n}}$, because the binomial coefficients are less than $\frac{2^n}{\sqrt{n}}$. Take the i for which p_i is the lowest. Define the distribution $D(j) = \binom{n}{l}$ where l is the greatest integer $\leq j$ and $l = i \bmod k$. This distribution lies in between the two distributions mentioned above. Thus the sum of absolute value of the terms which are negative is at most $\frac{k}{\sqrt{n}}$. A similar argument bounds the sum of the positive terms. □

Before we get to the main part of the induction we must choose our parameters and make another definition. Choose h_n so that

$$\sqrt{h_n} = (l(n-1))(n^{10})^{(n^{10}-1)},$$

the length of the longest n region. Define $\epsilon_2 = 1/(2^{h_2} + 1)$ and $\epsilon_n = \epsilon_{n-1}(1 - 100n^{-2})$. As $\sum 100n^{-2} < \infty$, $\lim \epsilon_n = \epsilon > 0$.

There is one last definition that we will use heavily. We say that a tree of height h over a point y **falls mostly in an n block** (or n region) if the interval $(h/2 - n\sqrt{h}, h/2 + n\sqrt{h})$ is entirely in one n block (region) in y. If an h_{n-1} subtree of the h_n tree over y or y' falls mostly in B^m_{n-1}, for some m, then label that subtree m. An application of lemma 4.2 shows that most h_{n-1} subtrees of a k_n subtree over any point fall mostly in one $n-1$ block and thus receive a label. This fraction is at least

$$1-\frac{2l(n-1)}{\sqrt{k_n}}-\frac{2n\sqrt{h_{n-1}}}{l(n-1)} \quad > \quad 1-4n^{-2}.$$

Now choose $k_n^{i,j}=k_n$ so that

$$\sqrt{k_n}=l(n-1)(n^{10})^{i-1/2}.$$

This length $\sqrt{k_n}$ is between the length of the n regions in B_n^i and the length of the n region in B_n^j. This will imply that most k_n subtrees in the h_n tree over y have many different labels on their h_{n-1} subtrees. But most k_n subtrees in the h_n tree over y' have essentially only one type of labeled h_{n-1} subtree. The next lemma will prove the first of these assertions and the second lemma will prove the latter. The combination of the two will form the heart of the inductive step.

Lemma 4.3. *If the h_n tree over y falls mostly over n block B_n^i, then for a fraction $1-2n^{-2}$ of the k_n subtrees in the h_n tree over y, and for any given label, a fraction at most $3n^{-2}$ of the h_{n-1} subtrees have that label.*

Proof. We will use an estimate on the binomial coefficients to show that k_n subtrees of the h_n tree over y which fall mostly in n block B_n^i have many different labels on their h_{n-1} subtrees. None of these labels have a large fraction of the h_{n-1} subtrees with that label. First we show that only a small fraction of the branches of a k_n tree land in any given $n-1$ region in n block B_n^i. Then we use that to show that only a small fraction of h_{n-1} blocks in a k_n tree can have the same label. No binomial coefficient is larger than $2^n/\sqrt{n}$. So the number of branches of a k_n tree that land in a given h_{n-1} region is at most $2^{k_n}/\sqrt{k_n}$ multiplied by the length of an n region in block B_n^i. Since $\sqrt{k_n}$ is much larger than $l(n-1)(n^{10})^{i-1}$, the length of an n region in block B_n^i, the fraction of branches that land in a given h_{n-1} region is less than n^{-2}.

The distance between two h_{n-1} regions in block B_n^i that are made out of the same $n-1$ block is $l(n-1)(n^{10})^{i-1}(n^{10}-1)$. Since this is far larger than $\sqrt{k_n}$ a Chebychev estimate shows that if the number of branches in one h_{n-1} region has close to the bound then the number of branches that land in all other h_{n-1} regions that are made out of the same $n-1$ block is miniscule. So the fraction of branches that land in a given h_{n-1} block is less than $2n^{-2}$. The fraction of h_{n-1} subtrees with a given label is then less than $3n^{-2}$. Finally a Chebychev estimate shows that the fraction of k_n subtrees to which the preceding argument applies, those which fall mostly in n block B_n^i, is at least $1 - 2n^{-2}$ of all the k_n subtrees. □

For the k_n subtrees over y' the picture is completely different. Most k_n subtrees of the h_n tree over y' have the same label on almost all of their h_{n-1} subtrees.

Lemma 4.4. *If the h_n tree over y' falls mostly over n block B_n^j then for a fraction $1 - 6n^{-2}$ of the k_n subtrees in the h_n tree over y' a fraction at least $1 - 2n^{-2}$ of the h_{n-1} subtrees in one of these k_n subtrees have the same label.*

Proof. As in the last lemma the k_n subtrees which fall mostly in n block B_n^j, is at least $1 - 2n^{-2}$ of all the k_n subtrees. Of these subtrees, lemma 4.2 implies those that do fall mostly in an n region in block B_n^j make up a fraction at least

$$1 - \frac{2n\sqrt{k_n}}{l(n-1)(n^{10})^{j-1}} - \frac{2l(n-1)(n^{10})^{j-1}}{\sqrt{h_n}} > 1 - 4n^{-2}.$$

In such a k_n subtree, a Chebychev estimate shows that a fraction at least $1-2n^{-2}$ of the h_{n-1} subtrees have the same label. Thus in $1-6n^{-2}$ of the k_n subtrees in the big tree over y' a fraction at least $1 - 2n^{-2}$ of the h_{n-1} subtrees have the same label. □

Theorem 4.1. *The [T,Id] endomorphism is nonstandard.*

Proof. We will prove the following inductive statement. If the h_n tree over y falls mostly in n block B_n^i, the h_n tree over y' falls mostly in n block B_n^j, and $i \neq j$ then $v_{h_n}(y, y') > \epsilon_n$. This will complete the proof because $2n\sqrt{h_n}$ is much smaller than $l(n)$ so most points y have the h_n over y tree falling mostly in one n block. In addition $h_n \to \infty$, and $\epsilon_n \to \epsilon > 0$. Thus these facts, the induction, and theorem 2.1 will imply that [T,Id] is nonstandard.

Case $n = 2$: Since $\epsilon_2 = 1/(2^{h_2}+1)$ the two trees must match perfectly after some tree isomorphism. No word of length h_2 in one 2 block is identical to a word of length h_2 or its reflection in a different 2 block, or is of period 2, so lemma 4.1 shows the base case is true.

Case n: Assume the lemma is true for $n - 1$ and $i < j$, are given. A tree automorphism must take k_n subtrees to k_n subtrees, and h_{n-1} subtrees to h_{n-1} subtrees. By lemma 4.4 the k_n subtrees over y' have essentially only one type of labeled h_{n-1} subtree. But by lemma 4.3 the k_n subtrees over y have many different labels. Since most h_{n-1} subtrees are labeled, most h_{n-1} subtrees must be mapped to an h_{n-1} subtree with a different label. On these h_{n-1} subtrees the inductive hypothesis applies. Thus for any automorphism of the tree over y we can apply the inductive hypothesis to a fraction at least $1 - 100n^{-2}$ of the h_{n-1} subtrees. As $\epsilon_n = \epsilon_{n-1}(1 - 100n^{-2})$ the inductive step is true. This completes the proof. □

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