Minimal Surfaces and Bernstein's Theorem

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1 Introduction

The theory of minimal surfaces was originally motivated by the notion of surfaces that minimizes area given a fixed boundary. Such a surface in fact has zero mean curvature, although there are surfaces with zero mean curvature that do not minimize the area functional. Much of the results we will draw upon can be found in [1] and [3].

The focus of this paper will be to present a brief introduction to two-dimensional minimal surfaces embedded in $\mathbb{R}^n$. We will conclude by presenting an important theorem regarding solutions to the minimal surface equation in the entire plane, called Bernstein's Theorem. An important corollary to this main theorem can be summarized as follows: If $f : \mathbb{R}^2 \to \mathbb{R}$ is a non-parametric representation of a surface S that satisfies the minimal surface equation in the whole x_1, x_2-plane, then f is necessarily a linear function of x_1 and x_2. In other words, S is necessarily a plane.

In Section 1, some relevant background and notation is established, and for the most part it is intended as a general review of regular surfaces. Section 2 introduces the minimal surface equation, and in Section 3 we introduce the notion of nonparametric surfaces embedded in $\mathbb{R}^n$ and derive the minimal surface equation applied to them. Then in Section 4, isothermal parameters are described, and we then culminate with a proof that every minimal surface can be reparametrized locally with respect to isothermal coordinates. Finally, Section 5 is a presentation of Bernstein's Theorem and some important corollaries.

2 Some Background

A two-dimensional surface $S \subset \mathbb{R}^n$ by definition can be parametrized locally near each point by a smooth map $x : D \to S$ where $D \subset \mathbb{R}^2$ is open, and such that the vectors $\frac{\partial x}{\partial u_1}$ and $\frac{\partial x}{\partial u_2}$ are linearly independent at each point of D. We use M to denote the Jacobian matrix of x and x_i, $1 \leq i \leq n$, to denote the coordinate functions of x, so that

$$M = \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_1} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \\ \vdots & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} \end{bmatrix}$$

In shorthand, we may write $M = (m_{ij})$ where $m_{ij} = \frac{\partial x_i}{\partial u_j}$.

Given vectors $v, w \in \mathbb{R}^3$, it is intuitive to think of $v \wedge w$, called the wedge product of v and w, as representing the vector in $\mathbb{R}^3$ perpendicular to both v and w, given that v is not parallel to w. This idea generalizes to vectors in $\mathbb{R}^n$ as follows: if $v, w \in \mathbb{R}^n$, then we define

$$v \wedge w \in \mathbb{R}^N,\ N = \binom{n}{2}$$

to be the vector with components

$$\det \begin{pmatrix} v_i & v_j \\ w_i & w_j \end{pmatrix},\ \ i < j,$$

arranged in some fixed order.

We now introduce the matrix G defined by

$$G = (g_{ij}) = M^T M. \tag{1}$$

More explicitly, we have

$$g_{ij} = \left\langle \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} \right\rangle = \sum_{k=1}^{n} \frac{\partial x_k}{\partial u_i} \frac{\partial x_k}{\partial u_j}.$$

Lemma 1. *Let $x : D \to \mathbb{R}^n$ be a differentiable map where $D \subset \mathbb{R}^2$ is a domain with coordinates (u_1, u_2). At each point of D, the following are equivalent:*

i. *the vectors $\frac{\partial x}{\partial u_1}$ and $\frac{\partial x}{\partial u_2}$ are linearly independent,*

ii. *the Jacobian matrix M has rank 2,*

iii. *$\exists i, j$ with $1 \leq i < j \leq n$ such that $\frac{\partial(x_i, x_j)}{\partial(u_1, u_2)} \neq 0$,*

iv. $\frac{\partial x}{\partial u_1} \wedge \frac{\partial x}{\partial u_2} \neq 0$,

v. $\det G > 0$.

Definition 1. A surface S is *regular* at a point if the conditions of Lemma 1 hold at that point. Similarly, S is regular if it is regular at every point of D.

Suppose we have a regular surface $S \subset \mathbb{R}^n$ defined by the parametrization $x : D \to S$, and let $p \in S$. Recall that the tangent space T_pS is a two-dimensional vector space that is spanned by all the tangent vectors at p attained by curves lying on S and passing through p. That is, $v \in T_pS$ if there exists a smooth curve $\alpha : (-\epsilon, \epsilon) \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = v$. We also denote the orthogonal complement of T_pS by $T_pS^{\perp}$, which is consequently an $(n-2)$-dimensional space called the normal space of S at p.

2.1 Preliminary Minimal Surface Equation

Let $S \subset \mathbb{R}^n$ be a regular surface defined by the parametrization $x : D \to S$ and let $p \in S$. If $\alpha : (-\epsilon, \epsilon) \to S$ is a smooth curve p.b.a.l. with $\alpha(0) = p$ and $\alpha'(0) = v$ with $|v| = 1$, then we define

$$k(N, v) = \alpha''(0) \cdot N, \quad N \in T_pS^{\perp}. \tag{2}$$

The function $k(N, v)$ is a well-defined function of the normal N and the unit tangent v, called the *normal curvature* of S in the direction of v. For more details concerning this derivation, see [Osserman].

By fixing N and letting v vary, we define

$$k_1(N) = \max_{v \in T_pS} k(N, v), \quad k_2(N) = \min_{v \in T_pS} k(N, v), \tag{3}$$

which are called the *principal curvatures* of S at p, with respect to the normal N. Finally, we define the *mean curvature* of S, $H(N)$, at the point p, with respect to N, by

$$H(N) = \frac{k_1(N) + k_2(N)}{2}. \tag{4}$$

With a bit of computation, (see Osserman or Ros), one obtains the identity

$$H(N) = \frac{g_{22} b_{11}(N) + g_{11} b_{22}(N) - g_{12} b_{12}(N)}{2 \det(G)}. \tag{5}$$

Here we have introduced the notation

$$b_{ij}(N) = \frac{\partial^2 x}{\partial u_i \partial u_j} \cdot N.$$

It is clear that the b_{ij} are linear in N, and hence $H(N)$ is linear in N for all $N \in T_p S^\perp$. Thus there exists a unique vector $H \in T_p S^\perp$ such that

$$H(N) = H \cdot N \;\; \forall N \in T_p S^\perp.$$

The vector H thus defined is called the *mean curvature vector* of S at the point p.

Definition 2. A surface S is a minimal surface if its mean curvature vector H vanishes at every point $p \in S$.

Equivalently, we may characterize minimal surfaces by the equation

$$g_{22}b_{11}(N) + g_{11}b_{22}(N) - g_{12}b_{12}(N) = 0. \tag{6}$$

3 Non-parametric Surfaces

We now consider a special choice of parameters that is often useful in computations. Let i, j be any two fixed distinct integers such that $1 \le i, j \le n$, and let D be a domain in the x_i, x_j plane. The equations

$$x_k = f_k(x_i, x_j), k = 1, \ldots, n; \quad k \neq i, j; \quad (x_i, x_j) \in D \tag{7}$$

define a surface S in $\mathbb{R}^n$. A surface defined in this way is said to be in *nonparametric* or *explicit* form. In the case that $n = 3$, the surface S is simply represented as a graph over the x_i, x_j plane. It is a well-known result that any surface can be expressed locally in nonparametric form, as stated in the following lemma. The proof is for the most part a consequence of the inverse function theorem (see [1], [3]).

Lemma 2. *Let $S \subset \mathbb{R}^n$ be a surface defined by a parametrization $x : D \to \mathbb{R}^n$, and let $p = x(q)$ be a regular point of S. Then there exists a neighborhood U of q, such that the restricted surface $x(U) \subset S$ has a reparametrization in nonparametric form.*

Again suppose we have a surface S in nonparametric form. By a suitable relabeling of coordinates in $\mathbb{R}^n$, we may assume that the surface S is defined by

$$x_k = f_k(x_1, x_2), \quad k = 3, \ldots, n,$$

or equivalently

$$x_1 = u_1, \quad x_2 = u_2, \quad x_k = f_k(u_1, u_2), \quad k = 3, \ldots, n. \tag{8}$$

With a bit of computation (see [1]), the minimal surface equation takes the form

$$\sum_{k=3}^{n} \left[\left(1 + \sum_{m=3}^{n} \left(\frac{\partial f_m}{\partial u_2}\right)^2\right) \frac{\partial^2 f_k}{\partial u_1^2} - 2\left(\sum_{m=3}^{n} \frac{\partial f_m}{\partial u_1}\frac{\partial f_m}{\partial u_2}\right) \frac{\partial^2 f_k}{\partial u_1 \partial u_2} \right.$$
$$\left. + \left(1 + \sum_{m=3}^{n} \left(\frac{\partial f_m}{\partial u_1}\right)^2\right) \frac{\partial^2 f_k}{\partial u_2^2} \right] N_k = 0,$$

for all normal vectors N. It is an elementary result ([Osserman]) that the components $N_3, \ldots, N_n$ may be chosen arbitrarily. Thus, each of the coefficients of N_k must vanish in the equation above, for $k = 3, \ldots, n$. Introducing the notation

$$f(x_1, x_2) = (f_3(x_1, x_2), \ldots, f_n(x_1, x_2)),$$

we have the single vector equation

$$\left(1 + \left|\frac{\partial f}{\partial x_2}\right|^2\right) \frac{\partial^2 f}{\partial x_1^2} - 2\left(\frac{\partial f}{\partial x_1} \cdot \frac{\partial f}{\partial x_2}\right) + \left(1 + \left|\frac{\partial f}{\partial x_1}\right|^2\right) \frac{\partial^2 f}{\partial x_2^2} = 0. \tag{9}$$

This is the minimal surface equation for nonparametric minimal surfaces in $\mathbb{R}^n$. In light of Lemma 2, every regular surface provides a local solution to (9).

4 Isothermal Parameters

Some properties of a surface, such as curvature and compactness, are independent of the choice of parametrization. Thus it is convenient to choose parameters in a way such that geometric properties of the surface are reflected in the plane. For example, it would be nice if the angles between curves in the domain D are preserved under the mapping $x : D \to S$. This is essentially the definition of a conformal mapping. Explicitly, this requirement can be expressed as

$$g_{11} = \left|\frac{\partial x}{\partial u_1}\right|^2 = \left|\frac{\partial x}{\partial u_2}\right|^2 = g_{22}, \qquad g_{12} = \langle \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_1} \rangle = 0,$$

or

$$G = \lambda^2 I_2$$

where $\lambda : D \to \mathbb{R}$ is a function such that $\lambda(u_1, u_2) > 0$. Parameters u_1, u_2 satisfying these conditions are called *isothermal parameters.*

Note that, when working with isothermal parameters, we have

$$\det G = \lambda^4 \tag{10}$$

and

$$\begin{aligned} H(N) &= \frac{g_{22}b_{11}(N) + g_{11}b_{22}(N) - g_{12}b_{12}(N)}{2\det(G)} \\ &= \frac{\lambda^2 b_{11}(N) + \lambda^2 b_{22}(N)}{\lambda^4} \\ &= \frac{b_{11}(N) + b_{22}(N)}{\lambda^2}. \end{aligned} \tag{11}$$

It turns out, as might be expected, that a parametrization of a minimal surface in terms of isothermal parameters is necessarily a harmonic function. This is expressed in the following lemma and its immediate corollary.

Lemma 3 (Osserman)**.** *Let $S \subset \mathbb{R}^n$ be a regular surface parametrized by $x(u_1, u_2)$ where u_1 and u_2 are isothermal parameters. Then*

$$\Delta x = 2\lambda^2 H$$

where H is the mean curvature vector.

Corollary 4. *Let $x(u_1, u_2)$ define a regular surface S in isothermal parameters. Then the coordinate functions $x_k(u_1, u_2)$ are harmonic if and only if S is a minimal surface.*

4.1 Minimal Surfaces and Analytic Functions

It will eventually be useful to have some important ideas from complex analysis at our disposal. We recall the following definitions.

Definition 3. A function $f(z) = u(x, y) + iv(x, y)$ is *conformal* on an open set $D \subset \mathbb{C}$ if f is analytic on D and $f'(z) \neq 0$ for all $z \in D$.

Definition 4. A function $f(z) = u(x, y) + iv(x, y)$ is *anticonformal* on an open set D, if the conjugate function $\overline{f(z)} = u(x, y) - iv(x, y)$ is conformal.

In light of Lemma 3, it will be interesting to look further into the connection between minimal surfaces and harmonic functions. First, consider the following notation. Given a surface S parametrized by $x : D \to S$, we define the complex-valued functions

$$\phi_k(\xi) = \frac{\partial x_k}{\partial u_1}(u_1, u_2) - i\frac{\partial x_k}{\partial u_2}(u_1, u_2), \quad \xi = u_1 + iu_2. \tag{12}$$

Now consider the following identities derived from the ϕ_k:

$$\begin{aligned}
\sum_{k=1}^{n} \phi_k^2(\xi) &= \sum_{k=1}^{n} \left[\left(\frac{\partial x_k}{\partial u_1}\right)^2 - \left(\frac{\partial x_k}{\partial u_2}\right)^2 - 2i\frac{\partial x_k}{\partial u_1}\frac{\partial x_k}{\partial u_2} \right] \\
&= \sum_{k=1}^{n} \left(\frac{\partial x_k}{\partial u_1}\right)^2 - \sum_{k=1}^{n} \left(\frac{\partial x_k}{\partial u_2}\right)^2 - 2i\sum_{k=1}^{n} \frac{\partial x_k}{\partial u_1}\frac{\partial x_k}{\partial u_2} \\
&= \left|\frac{\partial x}{\partial u_1}\right|^2 - \left|\frac{\partial x}{\partial u_2}\right|^2 - 2i\left\langle \frac{\partial x}{\partial u_2}, \frac{\partial x}{\partial u_2} \right\rangle \\
&= g_{11} - g_{22} - 2ig_{12}
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
\sum_{k=1}^{n} |\phi_k(\xi)|^2 &= \sum_{k=1}^{n} \left(\frac{\partial x_k}{\partial u_1} - i\frac{\partial x_k}{\partial u_2}\right)\left(\frac{\partial x_k}{\partial u_1} + i\frac{\partial x_k}{\partial u_2}\right) \\
&= \sum_{k=1}^{n} \left(\frac{\partial x_k}{\partial u_1}\right)^2 + \sum_{k=1}^{n} \left(\frac{\partial x_k}{\partial u_2}\right)^2 \\
&= g_{11} + g_{22}.
\end{aligned} \tag{14}$$

Based on these equations, we can formulate the following useful proposition regarding the functions ϕ_k:

Proposition 5. *Suppose that S is a surface parametrized by $x : D \to S$. If the functions $\phi_k(\xi)$ are defined according to (12), then the following statements hold:*

i. $\phi_k(\xi)$ is analytic in ξ if and only if x_k is harmonic in u_1, u_2,

ii. u_1, u_2 are isothermal parameters if and only if

$$\sum_{k=1}^{n} \phi_k^2(\xi) = 0, \tag{15}$$

iii. If u_1, u_2 are isothermal parameters, then S is regular if and only if

$$\sum_{k=1}^{n} |\phi_k(\xi)|^2 \neq 0. \tag{16}$$

Proof. For (i), note that $\phi_k(\xi)$ is analytic in $\xi \iff \frac{\partial^2 x_k}{\partial u_1^2} = \frac{\partial^2 x_k}{\partial u_2^2} \iff x_k$ is harmonic in u_1, u_2.

To prove (ii), we have that u_1, u_2 are isothermal parameters $\iff g_{11} = g_{22}$ and $g_{12} = 0$ $\iff \sum_{k=1}^{n} \phi_k^2(\xi) = 0$.

Finally, for (iii) note that u_1, u_2 are isothermal parameters if and only if $g_{11} = g_{22}$ and $g_{12} = 0$. Furthermore, S is regular $\iff \det G \neq 0 \iff g_{11}g_{22} = g_{11}^2 \neq 0 \iff \sum_{k=1}^{n} |\phi_k(\xi)|^2 = g_{11} + g_{22} \neq 0$ since g_{11} and g_{22} are both nonnegative. □

It turns out that one can choose any analytic functions $\phi_k(\xi)$ that satisfy (13) and (14) in a simply-connected domain D, and there necessarily exists a regular minimal surface defined by $x : D \to \mathbb{R}^3$ such that (12) holds. For a proof of this, see [1]. Thus, there is a very strong connection indeed between minimal surfaces and complex-valued analytic functions.

Lemma 6. *Let $x(u)$ define a regular minimal surface with u_1, u_2 isothermal parameters. Then the functions $\phi_k(\xi)$ defined by (12) are analytic, and they satisfy equations (15) and (16). Conversely, let $\phi_1(\xi), \ldots, \phi_n(\xi)$ be analytic functions of ξ which satisfy (15) and (16) in a simply-connected domain D. Then there exists a minimal surface $x(u)$ defined over D, such that equations (12) hold.*

Another very important result regarding isothermal parameters is analogous to the fact that any parametrization $x : D \to S$ can be rewritten in nonparametric form locally. It is worth proving, as the notation used will be revisited in the proof of Bernstein's Theorem.

Lemma 7. *Let S be a minimal surface. Then every regular point of S has a neighborhood in which there exists a reparametrization of S in terms of isothermal parameters.*

Proof. Let $p \in S$ be a point at which S is regular. Then we may find a neighborhood V of p in which S may be represented in nonparametric form. That is, there exists an open set $D \subset \mathbb{R}^2$ and a diffeomorphism $x : D \to V$ defined by $x_k = f_k(x_1, x_2)$ for $k = 3, \ldots, n$, where f is a smooth function from D to $\mathbb{R}^{n-2}$. We now introduce the notation

$$\begin{aligned} f = (f_3, \ldots, f_n), \ \ p = \frac{\partial f}{\partial x_1}, \ \ q = \frac{\partial f}{\partial x_2}, \\ r = \frac{\partial^2 f}{\partial x_1^2}, \ \ s = \frac{\partial^2 f}{\partial x_1 \partial x_2}, \ \ t = \frac{\partial^2 f}{\partial x_2^2}. \end{aligned} \tag{17}$$

We also write $W = \det G$. It follows from the minimal surface equation (9) (see [Osserman]) that

$$\begin{aligned} \frac{\partial}{\partial x_1}\left(\frac{1+|q|^2}{W}\right) = \frac{\partial}{\partial x_2}\left(\frac{p \cdot q}{W}\right) \\ \frac{\partial}{\partial x_1}\left(\frac{p \cdot q}{W}\right) = \frac{\partial}{\partial x_2}\left(\frac{1+|p|^2}{W}\right). \end{aligned} \tag{18}$$

Suppose that $p = x(a_1, a_2)$. Then by openness of D we have equations (18) satisfied in some disk $(x_1 - a_1)^2 + (x_2 - a_2)^2 < R^2$, which we'll denote by D'. If we define the vector field V in $\mathbb{R}^3$ by

$$V_1 = \left(\frac{1 + |p|^2}{W}, \frac{p \cdot q}{W}, 0\right),$$

then

$$|\nabla \times V_1| = \frac{\partial}{\partial x_2}\left(\frac{1 + |p|^2}{W}\right) - \frac{\partial}{\partial x_1}\left(\frac{p \cdot q}{W}\right) = 0.$$

Hence V_1 is a conservative vector field, and so there exists a function $F : D' \to \mathbb{R}$ such that $\nabla F = V_1$. Similarly, the vector field

$$V_2 = \left(\frac{p \cdot q}{W}, \frac{1 + |q|^2}{W}, 0\right)$$

is conservative, and so there exists a function $G : D' \to \mathbb{R}$ such that $\nabla G = V_2$. Altogether, the functions $F(x_1, x_2)$ and $G(x_1, x_2)$ satisfy

$$\begin{aligned} \frac{\partial F}{\partial x_1} &= \frac{1 + |p|^2}{W}, \quad \frac{\partial F}{\partial x_2} = \frac{p \cdot q}{W} \\ \frac{\partial G}{\partial x_1} &= \frac{p \cdot q}{W}, \quad \frac{\partial G}{\partial x_2} = \frac{1 + |q|^2}{W}. \end{aligned} \tag{19}$$

Now let

$$\xi = (\xi_1, \xi_2), \;\; \xi_1 = x_1 + F(x_1, x_2), \;\; \xi_2 = x_2 + G(x_1, x_2), \tag{20}$$

and note that

$$\begin{aligned} \frac{\partial \xi_1}{\partial x_1} &= 1 + \frac{1 + |p|^2}{W} \quad \frac{\partial \xi_1}{\partial x_2} = \frac{p \cdot q}{W} \\ \frac{\partial \xi_2}{\partial x_1} &= \frac{p \cdot q}{W} \quad \frac{\partial \xi_2}{\partial x_2} = 1 + \frac{1 + |q|^2}{W}. \end{aligned}$$

Additionally, one can compute

$$\begin{aligned} J &= \det\begin{pmatrix} \frac{\partial \xi_1}{\partial x_1} & \frac{\partial \xi_1}{\partial x_2} \\ \frac{\partial \xi_2}{\partial x_1} & \frac{\partial \xi_2}{\partial x_2} \end{pmatrix} \\ &= 2 + \frac{2 + |p|^2 + |q|^2}{W} > 0. \end{aligned}$$

By the inverse function theorem, there exist open sets $U_1 \subset \mathbb{R}^2$ and $U_2 \subset D'$ where $(a_1, a_2) \in U_2$, and a diffeomorphism $\beta : U_1 \to U_2$ such that

$$(\beta \circ \xi)(x_1, x_2) = \beta(\xi_1, \xi_2) = (x_1, x_2)$$

for all $(x_1, x_2) \in U_2$. In the neighborhood U_1, we may represent the surface S locally in terms of the parameters ξ_1, ξ_2.

By the chain rule, we have $(d\beta)_{(\xi_1,\xi_2)}(d\xi)_{(x_1,x_2)} = \mathrm{Id} \Rightarrow (d\beta)_{(\xi_1,\xi_2)} = \left[(d\xi)_{(x_1,x_2)}\right]^{-1}$. Hence

$$\begin{aligned} (d\beta)_{(\xi_1,\xi_2)} &= \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix} \\ &= \frac{1}{J}\begin{bmatrix} \frac{\partial \xi_2}{\partial x_2} & -\frac{\partial \xi_1}{\partial \xi_2} \\ -\frac{\partial \xi_2}{\partial x_1} & \frac{\partial \xi_1}{\partial x_1} \end{bmatrix}. \end{aligned}$$

Equating components, we have

$$\frac{\partial x_1}{\partial \xi_1} = \frac{W+1+|q|^2}{JW}, \quad \frac{\partial x_2}{\partial \xi_1} = -\frac{p\cdot q}{JW},$$
$$\frac{\partial x_1}{\partial \xi_2} = -\frac{p\cdot q}{JW}, \quad \frac{\partial x_2}{\partial \xi_2} = \frac{W+1+|p|^2}{JW}.$$

Moreover, applying the chain rule to x_k for $k = 3,\dots,n$, we obtain

$$\begin{aligned}\frac{\partial x_k}{\partial \xi_1} &= \frac{\partial x_k}{\partial x_1}\frac{\partial x_1}{\partial \xi_1} + \frac{\partial x_k}{\partial x_2}\frac{\partial x_2}{\partial \xi_1} \\ &= \frac{W+1+|q|^2}{JW}p_k - \frac{p\cdot q}{JW}q_k \\ \frac{\partial x_k}{\partial \xi_2} &= \frac{\partial x_k}{\partial x_1}\frac{\partial x_1}{\partial \xi_2} + \frac{\partial x_k}{\partial x_2}\frac{\partial x_2}{\partial \xi_2} \\ &= \frac{W+1+|p|^2}{JW}q_k - \frac{p\cdot q}{JW}p_k.\end{aligned}$$

From all of these computations, we deduce that

$$\begin{aligned} g_{11} = g_{22} &= \left|\frac{\partial x}{\partial \xi_1}\right|^2 = \left|\frac{\partial x}{\partial \xi_2}\right|^2 = \frac{W}{J} = \frac{W^2}{2W+2+|p|^2+|q|^2} \\ g_{12} &= \frac{\partial x}{\partial \xi_1}\cdot\frac{\partial x}{\partial \xi_2} = 0.\end{aligned}$$

Hence ξ_1, ξ_2 are isothermal parameters. □

The next result provides an explicit connection between conformal or anticonformal mappings of the complex plane and surfaces defined in terms of isothermal parameters. The proof is more or less elementary, but we will need it when proving Bernstein's Theorem.

Lemma 8. *Let $S \subset \mathbb{R}^n$ be a surface defined by $x(u_1, u_2)$ where $x : D \subset \mathbb{R}^2 \to S$ and u_1, u_2 are isothermal parameters. Suppose that $\phi : \widetilde{D} \to D$ is a diffeomorphism given by*

$$\phi(\widetilde{u}_1, \widetilde{u}_2) = (u_1(\widetilde{u}_1, \widetilde{u}_2), u_2(\widetilde{u}_1, \widetilde{u}_2)),$$

and let $\widetilde{S}$ be the reparametrized surface obtained from the composition $x \circ \phi$. Then $\widetilde{u}_1$ and $\widetilde{u}_2$ are isothermal parameters if and only if the map $\phi(\widetilde{u}_1, \widetilde{u}_2)$ is either conformal or anticonformal.

Proof. Let U be the matrix defined by

$$U = \operatorname{Jac}\phi = \begin{bmatrix} \frac{\partial u_1}{\partial \widetilde{u}_1} & \frac{\partial u_1}{\partial \widetilde{u}_2} \\ \frac{\partial u_2}{\partial \widetilde{u}_1} & \frac{\partial u_2}{\partial \widetilde{u}_2} \end{bmatrix}.$$

Since ϕ is a diffeomorphism, we have $\det U \neq 0$. Let $\widetilde{M}$ be the Jacobian matrix of $x \circ \phi$. By the chain rule,

$$\widetilde{M} = MU,$$

where $M = \operatorname{Jac} x$, and thus

$$\widetilde{G} = \widetilde{M}^T\widetilde{M} = (MU)^T MU = U^T M^T MU = U^T GU.$$

Since u_1 and u_2 are isothermal parameters, we have that $G = \lambda^2 I_2$. Thus $\widetilde{G} = \lambda^2 U^T U$. By definition $\widetilde{u}_1, \widetilde{u}_2$ are isothermal if and only if $\widetilde{G} = \widetilde{\lambda}^2 I_2$. This in turn is the case if and only if $\frac{\lambda^2}{\widetilde{\lambda}^2} U^T U = I_2$; that is, if and only if $\frac{\lambda}{\widetilde{\lambda}} U$ is an orthogonal matrix.

Note that

$$U^T U = \begin{bmatrix} \left(\frac{\partial u_1}{\partial \widetilde{u}_1}\right)^2 + \left(\frac{\partial u_1}{\partial \widetilde{u}_2}\right)^2 & \frac{\partial u_1}{\partial \widetilde{u}_1}\frac{\partial u_2}{\partial \widetilde{u}_1} + \frac{\partial u_1}{\partial \widetilde{u}_2}\frac{\partial u_2}{\partial \widetilde{u}_2} \\ \frac{\partial u_1}{\partial \widetilde{u}_1}\frac{\partial u_2}{\partial \widetilde{u}_1} + \frac{\partial u_1}{\partial \widetilde{u}_2}\frac{\partial u_2}{\partial \widetilde{u}_2} & \left(\frac{\partial u_2}{\partial \widetilde{u}_1}\right)^2 + \left(\frac{\partial u_2}{\partial \widetilde{u}_2}\right)^2 \end{bmatrix}$$

Supposing that $\frac{\lambda}{\widetilde{\lambda}} U$ is orthogonal, we have that

$$\frac{\partial u_1}{\partial \widetilde{u}_1}\frac{\partial u_2}{\partial \widetilde{u}_1} + \frac{\partial u_1}{\partial \widetilde{u}_2}\frac{\partial u_2}{\partial \widetilde{u}_2} = 0, \qquad \left(\frac{\partial u_1}{\partial \widetilde{u}_1}\right)^2 + \left(\frac{\partial u_1}{\partial \widetilde{u}_2}\right)^2 = \left(\frac{\partial u_2}{\partial \widetilde{u}_1}\right)^2 + \left(\frac{\partial u_2}{\partial \widetilde{u}_2}\right)^2.$$

Let $a = \frac{\partial u_1}{\partial \widetilde{u}_1}$, $b = \frac{\partial u_1}{\partial \widetilde{u}_2}$, $c = \frac{\partial u_2}{\partial \widetilde{u}_1}$, and $d = \frac{\partial u_2}{\partial \widetilde{u}_2}$. Then

$$a^2 + b^2 = c^2 + d^2, \qquad ac + bd = 0.$$

One of a and b is nonzero, since otherwise $U = 0$. Assume that $a \neq 0$. Then $c = -\frac{bd}{a}$, and substituting into the first equation yields

$$\begin{aligned} &a^2 + b^2 = \frac{b^2 d^2}{a^2} + d^2 \\ \Rightarrow &a^4 + b^2 a^2 = b^2 d^2 + d^2 a^2 \\ \Rightarrow &a^4 + (b^2 - d^2)a^2 - b^2 d^2 = 0. \end{aligned}$$

It follows that

$$\begin{aligned} a^2 &= \frac{d^2 - b^2}{2} \pm \frac{\sqrt{(b^2 - d^2)^2 + 4b^2 d^2}}{2} \\ &= \frac{d^2 - b^2}{2} \pm \frac{\sqrt{(b^2 + d^2)^2}}{2} \\ &= \frac{d^2 - b^2}{2} \pm \frac{b^2 + d^2}{2} \\ &= d^2, -b^2 \end{aligned}$$

Since a is real-valued, we must have $a^2 = d^2$, and thus $a = \pm d$. If $a = d$, then $b = -c$, and so

$$\frac{\partial u_1}{\partial \widetilde{u}_1} = \frac{\partial u_2}{\partial \widetilde{u}_2}, \quad \frac{\partial u_1}{\partial \widetilde{u}_2} = -\frac{\partial u_2}{\partial \widetilde{u}_1},$$

which means that ϕ is analytic in $\widetilde{u}_1$ and $\widetilde{u}_2$, when we consider it as a complex-valued function of $\widetilde{u}_1 + i\widetilde{u}_2$ with real part u_1 and imaginary part u_2. Hence ϕ is conformal.

On the other hand, if $a = -d$, then $b = c$, and it is easy to see in this case that ϕ is anticonformal.

Conversely, suppose that ϕ is either conformal or anticonformal. Then either

$$\frac{\partial u_1}{\partial \widetilde{u}_1} = \frac{\partial u_2}{\partial \widetilde{u}_2}, \quad \frac{\partial u_1}{\partial \widetilde{u}_2} = -\frac{\partial u_2}{\partial \widetilde{u}_1}$$

or

$$\frac{\partial u_1}{\partial \widetilde{u}_1} = -\frac{\partial u_2}{\partial \widetilde{u}_2}, \quad \frac{\partial u_1}{\partial \widetilde{u}_2} = \frac{\partial u_2}{\partial \widetilde{u}_1}.$$

In both cases, we conclude that $U^T U = \frac{\widetilde{\lambda}^2}{\lambda^2} I_2$ where $\widetilde{\lambda}$ is some function of $\widetilde{u}_1$ and $\widetilde{u}_2$. Since $\widetilde{G} = \lambda^2 U^T U = \widetilde{\lambda}^2 I_2$, it is clear that $\frac{\lambda}{\widetilde{\lambda}} U$ is an orthogonal matrix. $\square$

5 Bernstein's Theorem

We are now essentially ready to prove Bernstein's Theorem, the main result of this paper. But first, it will be necessary to present a couple of short lemmas. Additionally, the next theorem, of which a proof can be found in most books in complex analysis, will also be instrumental in proving the main theorem.

Theorem 9. *(Little Picard's Theorem) If a function $f : \mathbb{C} \to \mathbb{C}$ is nonconstant and entire, then the range of f omits at most a single point.*

Lemma 10. *Let $f(x_1, x_2)$ be a solution of (9) for $x_1^2 + x_2^2 < R^2$. Then using the notation of (17) and (19), the map $\xi(x_1, x_2)$ defined by (20) is a diffeomorphism onto a domain Δ that contains a disk of radius R centered at $\xi(0, 0)$.*

For a proof of the Lemma 10, see [1] (Lemma 5.4). We now prove a nice result that applies to Bernstein's Theorem in the case that $n = 3$.

Lemma 11. *Let $f : D \to \mathbb{R}$ be a smooth function on a domain $D \subset \mathbb{R}^2$. Let $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = f(x_1, x_2)\}$ be the surface obtained from f. Then S lies on a plane if and only if there exists a nonsingular linear transformation $(u_1, u_2) \to (x_1, x_2)$ such that u_1, u_2 are isothermal parameters.*

Proof. Suppose that such isothermal parameters u_1, u_2 exist, so that $x_1 = a_1u_1 + a_2u_2$ and $x_2 = b_1u_1 + b_2u_2$. Introducing the functions $\phi_k(\xi)$ from (12) for $k = 1, 2, 3$, it is clear that ϕ_1 and ϕ_2 are constant since x_1 and x_2 are linear functions of u_1 and u_2. By Proposition 5, we have that

$$\sum_{k=1}^{3} \phi_k^2(\xi) = 0,$$

and thus ϕ_3 must also be constant. This means that x_3 has constant gradient with respect to u_1 and u_2, and hence also with respect to x_1 and x_2. Thus $f(x_1, x_2) = Ax_1 + Bx_2 + C$.

Conversely, if S lies on a plane, then f is of the form $f(x_1, x_2) = Ax_1 + Bx_2 + C$. It is then easy to write down an explicit linear transformation yielding isothermal coordinates. For example, take

$$x_1 = \lambda Au_1 + Bu_2, \quad x_2 = \lambda Bu_1 - Au_2,$$

where

$$\lambda^2 = \frac{1}{1 + A^2 + B^2}.$$

□

Theorem 12. *(Bernstein's Theorem) Let $f(x_1, x_2)$ be a solution of the nonparametric minimal surface equation in the entire x_1, x_2 plane. Then there exists a nonsingular linear transformation*

$$\begin{aligned} x_1 &= u_1 \\ x_2 &= au_1 + bu_2, \quad b > 0, \end{aligned} \tag{21}$$

such that (u_1, u_2) are global isothermal parameters for the surface S defined by

$$x_k = f_k(x_1, x_2), \quad k = 3, \ldots, n.$$

Proof. First we consider the map ξ given in (20), which is now defined in the entire x_1, x_2 plane. By Lemma 10, the map ξ is a diffeomorphism of the x_1, x_2 plane onto the entire ξ_1, ξ_2 plane. By construction, we have that (ξ_1, ξ_2) are isothermal parameters of the surface S. By Lemma 6, the functions

$$\phi_k(\xi) = \frac{\partial x_k}{\partial \xi_1} - i\frac{\partial x_k}{\partial \xi_2}, \qquad k = 1, \dots, n$$

are analytic functions of ξ. Now observe that

$$\begin{aligned}\overline{\phi}_1\phi_2 &= \left(\frac{\partial x_1}{\partial \xi_1} + i\frac{\partial x_1}{\partial \xi_2}\right)\left(\frac{\partial x_2}{\partial \xi_1} - i\frac{\partial x_2}{\partial \xi_2}\right) \\ &= \frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_1} - i\frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2} + i\frac{\partial x_1}{\partial \xi_2}\frac{\partial x_2}{\partial \xi_1} + \frac{\partial x_1}{\partial \xi_2}\frac{\partial x_2}{\partial \xi_2}.\end{aligned}$$

Thus

$$\operatorname{Im}(\overline{\phi}_1\phi_2) = \frac{\partial x_1}{\partial \xi_2}\frac{\partial x_2}{\partial \xi_1} - \frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2} = -\frac{\partial(x_1, x_2)}{\partial(\xi_1, \xi_2)}.$$

Since the Jacobian determinant on the right is always positive, we have that $\phi_1 \neq 0$ and $\phi_2 \neq 0$ everywhere. Furthermore,

$$\begin{aligned}\operatorname{Im}\left(\frac{\phi_2}{\phi_1}\right) &= \operatorname{Im}\left(\frac{\phi_2\overline{\phi}_1}{|\phi_1|^2}\right) \\ &= \frac{1}{|\phi_1|^2}\operatorname{Im}(\overline{\phi}_1\phi_2) \\ &< 0.\end{aligned}$$

Therefore, the quotient $\frac{\phi_2}{\phi_1}$ is analytic on the whole ξ plane with negative imaginary part. By Picard's Theorem, $\frac{\phi_2}{\phi_1} = c$ where $c = a - bi$ and $b > 0$. This of course implies that $\phi_2 = c\phi_1$. By definition of ϕ_k, we have

$$\begin{aligned}\phi_2 = \frac{\partial x_2}{\partial \xi_1} - i\frac{\partial x_2}{\partial \xi_2} &= (a - bi)\left(\frac{\partial x_1}{\partial \xi_1} - i\frac{\partial x_1}{\partial \xi_2}\right) \\ &= a\frac{\partial x_1}{\partial \xi_1} - b\frac{\partial x_1}{\partial \xi_2} - i\left(a\frac{\partial x_1}{\partial \xi_2} + b\frac{\partial x_1}{\partial \xi_1}\right).\end{aligned}$$

Equating the real and imaginary parts from above, we obtain

$$\begin{aligned}\frac{\partial x_2}{\partial \xi_1} &= a\frac{\partial x_1}{\partial \xi_1} - b\frac{\partial x_1}{\partial \xi_2} \\ \frac{\partial x_2}{\partial \xi_2} &= a\frac{\partial x_1}{\partial \xi_2} + b\frac{\partial x_1}{\partial \xi_1}.\end{aligned}$$

If we now introduce the transformation of (21) using the values of a and b determined above, we compute

$$\begin{aligned}&\frac{\partial x_1}{\partial \xi_1} = \frac{\partial u_1}{\partial \xi_1}, &&\frac{\partial x_1}{\partial \xi_2} = \frac{\partial u_1}{\partial \xi_2} \\ &\frac{\partial x_2}{\partial \xi_1} = a\frac{\partial u_1}{\partial \xi_1} + b\frac{\partial u_2}{\partial \xi_1}, &&\frac{\partial x_2}{\partial \xi_2} = a\frac{\partial u_1}{\partial \xi_2} + b\frac{\partial u_2}{\partial \xi_2}.\end{aligned}$$

By substitution, it follows that

$$\frac{\partial x_2}{\partial \xi_2} = a\frac{\partial u_1}{\partial \xi_2} + b\frac{\partial u_2}{\partial \xi_2} = a\frac{\partial u_1}{\partial \xi_2} + b\frac{\partial u_1}{\partial \xi_1},$$

and so

$$\frac{\partial u_2}{\partial \xi_2} = \frac{\partial u_1}{\partial \xi_1}.$$

Similarly,

$$\frac{\partial u_2}{\partial \xi_1} = -\frac{\partial u_1}{\partial \xi_2}.$$

Thus, the function $u = (u_1, u_2)$ satisfies the Cauchy-Riemann equations with respect to ξ_1, ξ_2, and so the function $u_1 + iu_2$ is a complex-analytic function of $\xi_1 + i\xi_2$. By Lemma 8 u_1, u_2 are also isothermal parameters. □

As an immediate corollary, if we let $n = 3$, the solutions to the minimal surface equation (9) in the entire plane are limited to just linear functions.

Corollary 13. *In the case $n = 3$, the only solution of the minimal surface equation in the whole x_1, x_2 plane is the trivial solution, f a linear function of x_1, x_2.*

Proof. By Theorem 12, there exists a nonsingular linear transformation $x_1 = u_1$ and $x_2 = au_1 + bu_2$ with $b > 0$, such that u_1, u_2 are global isothermal parameters for the surface S defined by $x_3 = f(x_1, x_2)$. By Lemma 11, S lies in a plane, and hence f is a linear function. □

Corollary 14. *A bounded solution of equation (9) in the whole plane must be constant (for arbitrary n).*

Proof. By Lemma 4, each coordinate function $x_k(u_1, u_2)$, for $k = 3, \ldots, n$, is a bounded harmonic function of (u_1, u_2) in the whole u_1, u_2 plane. Hence, each x_k is constant for $3 \leq k \leq n$. □

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