INTRODUCTION TO DIFFERENTIAL EQUATIONS

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Lecture 11

2. Second order linear differential equations

We continue with our study in solving the differential equation

$$ay'' + by' + cy = 0. \tag{2.9}$$

We have divided our study into three cases, namely, (i) the characteristic equation has two different real roots, r_1 and r_2; (ii) has two conjugate complex roots, $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$; (iii) has a repeated real root. For the first case, we know the two special solutions are $e^{r_1 t}$ and $e^{r_2 t}$ and the general solution is provided by the linear combination of this two special solutions, namely

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

For the second case, the two special solution are $e^{\lambda t}\cos\mu t$ and $e^{\lambda t}\sin\mu t$. The general solution is provided again by the linear combination of the two special solutions

$$y = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\cos\mu t$$

with c_1 and c_2 two arbitrary (real) constants.

This gives us a feeling that to find the general solution of (2.9), one can first find two special solutions to (2.9), say y_1 and y_2, then make the linear combination to yield the general solution. Clearly, y_1 and y_2 cannot be arbitrary and they must satisfy certain condition. To find such condition, we need to introduce the so-called Wronskian of two functions $y_1(t)$ and $y_2(t)$:

$$W(y_1, y_2)(t) := \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \tag{2.10}$$

Now, let y_1 and y_2 be two solutions to (2.9), and we have the following criterion for $c_1 y_1 + c_1 y_2$ to be a general solution to (2.9).

Theorem 2.8. *If $W(y_1, y_2)(0) \neq 0$, then $c_1 y_1 + c_2 y_2$ is a general solution to (2.9).*

The proof is divided into two steps. In the first step, we need to show that $c_1y_1+c_2y_2$ is really a solution to (2.9), and this can be done by direct verification.

Next, we let $y=\phi(t)$ is a solution to (2.9). We show that there must exist some C_1 and C_2 such that $y=C_1y_1+C_2y_2$. We consider the auxiliary IVP

$$ay''+by'+cy=0,\quad y(0)=\phi(0),\quad y'(0)=\phi'(0). \tag{2.11}$$

Clearly, $y=\phi(t)$ is the unique solution to the above IVP. Next, we show that there exists C_1 and C_2 such that

$$\begin{cases} C_1y_1(0)+C_2y_2(0)=\phi(0), \\ C_1y_1'(0)+C_2y_2'(0)=\phi'(0). \end{cases}$$

In fact, the determinant of the coefficient matrix for the above linear system of equations is exactly the Wronskian of y_1 and y_2 at $t=0$. Since it is not zero, the above system is uniquely solvable. Hence, $y=C_1y_1+C_2y_2$ is also a solution to the IVP. By the fundamental theorem, we know that $\phi(t)=C_1y_1+C_2y_2$.

Now, one can check that $y=c_1e^{\lambda t}\cos\mu t+c_2e^{\lambda t}\sin\mu t$ is real "general" solution to the differential equation.

Example 2.9. $y''+9y=0$

Example 2.10. $16y''-8y'+145y=0,\quad y(0)=-2,\quad y'(0)=1.$

The solution is $y=-2e^{t/4}\cos 3t+1/2e^{t/4}\sin 3t$.

Lecture 12

Finally, we consider the case that $r_1=r_2=b/2a$. Clearly, we now have one solution given by

$$y_1=e^{bt/2a}.$$

We still need to find a second solution y_2 and to form the general solution with y_1 and y_2. Next, we will illustrate this by an example.

Example 2.11. $y''+4y'+4y=0.$

The solution $y_1=e^{-2t}$ is obvious. We want to find a different solution $y_2=v(t)y_1$. By straightforward calculations, one can show that

$$v''(t)=0,$$

and hence $y = (c_1 + c_2 t)e^{-2t}$ is a solution. It is clear now that $y_2 = te^{-2t}$ is a solution. One can verify that $W(y_1, y_2)(0) \neq 0$ and hence

$$y = c_1 e^{-2t} + c_2 t e^{-2t}$$

is a general solution.

This procedure can be extended to the general equation whose characteristic equation has repeated roots. And the general solution is given by

$$y = c_1 y_1 + c_2 t y_2$$

with

$$y_1 = e^{-bt/2a}, \qquad y_2 = te^{-bt/2a}.$$

Example 2.12. $y'' - y' + 0.25y = 0, \quad y(0) = 2, \quad y'(0) = 1/3.$

The solution is $y = 2e^{t/2} - 2/3te^{t/2}$.

Review of what we have learned up to now.

- Differential Equation (equation involving derivative)
  - 1. First order DE
    - 1. analytic solution
    - 2. qualitative analysis
    - 3. modeling
    - 4. numerical analysis
  - 2. Second order DE
    - 1. homogeneous linear eqn. with constant coefficients
    - 2. characteristic eqn.
    - 3. general solution

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