INTRODUCTION TO DIFFERENTIAL EQUATIONS

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Lecture 13

2. Second order linear differential equations

Section 3.6 in Textbook

So far, we have been solely concerned with the homogeneous second order differential equation. Next, we consider the inhomogeneous equation of the form

$$ay'' + by' + cy = g(t). \tag{2.12}$$

For the convenience of our study, we introduce the following notation,

$$L[y] = ay'' + by' + cy.$$

Then the equation can be written as

$$L[y](t) = g(t),$$

and the corresponding homogeneous equation is

$$L[y](t) = 0.$$

We next study the general structure of solutions to (2.12). To this end, we let $Y(t)$ be a particular solution to (2.12), and $y_1(t)$ and $y_2(t)$ be a set of fundamental solutions to the corresponding homogeneous equation.

Theorem 2.13. *For an arbitrary solution $Z(t)$ to (2.12), we have*

$$Z(t) = c_1 y_1 + c_2 y_2 + Y(t).$$

According to our theorem, in order to find the general solution of (2.12), we only need to find a particular solution. This is provided by the method of undetermined coefficients. The restriction for this method is that the inhomogeneous term $g(t)$ must take some special form.

Example 2.14.

$$y'' - 3y' - 4y = 3e^{2t}.$$

The particular solution is: $Y = -\frac{1}{2}e^{2t}$.

Example 2.15.

$$y'' - 3y' - 4y = -8e^t \cos 2t.$$

The particular solution is $Y(t) = Ae^t \cos 2t + Be^t \sin 2t$ with $A = -10/13$ and $B = 2/13$.

Now, we can see that in the method, one first assume that the particular solution takes some special form with some unknown coefficients, then using the differential equation to determine those coefficients. The crucial point here is how to choose the form the solution. This is dependent on what kind of form $g(t)$ is. We shall confine ourselves with some special $g(t)$'s. To this end, we introduce the following notation

$$P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_1 t + a_0.$$

We would consider that $g(t) = P_n(t)e^{\alpha t}$ or $g(t) = P_n(t)e^{\alpha t} \cos \beta t$ or $P_n(t)e^{\alpha t} \sin \beta t$

Lecture 14

We now give the general principle to choose the special form of particular solutions.

Case 1. $g(t) = P_n(t)e^{\alpha t}$

In such case, we will let

$$Y(t) = t^s (A_0 t^n + A_1 t^{n-1} + \cdots + A_{n-1} t^n + A_n)e^{\alpha t},$$

where s is the number of times that α is a root to the corresponding characteristic equation.

Example 2.16.

(1) $y'' - 5y' + 6y = (1 + t)e^{5t}$
(2) $y'' - 5y' + 6y = (1 + t)e^{2t}$
(3) $y'' - 2y' + y = te^t$

Case 2. $g(t) = P_n(t)e^{\alpha t} \cos \beta t$

In this case, we take the general solution to be of the following form

$$\begin{aligned} Y(t) =& t^s [(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)e^{\alpha t} \cos \beta t \\ &+ (B_0 t^n + B_1 t^{n-1} + \cdots + B_n)e^{\alpha t} \sin \beta t], \end{aligned}$$

where s is the number of times that $\alpha + \beta i$ is a root to the underlying characteristic equation.

Example 2.17.

(1) $y'' - 5y' + 6y = (1 + t)e^{5t} \sin t$
(2) $y'' - y' + y = (1 + t + t^2)e^{t/2} \cos \sqrt{3}t/2$

(3) $y'' - y' + y = te^t \sin \sqrt{3}t/2$

The same principle holds for the case with the RHS of the form $g(t) = P_n(t)e^{\alpha t} \cos \beta t$.

Case 2. $g(t) = g_1(t) + \cdots + g_n(t)$

Example 2.18.

$$y'' - 5y' + 6y = (1 + t)e^{5t} \sin t + (1 + t)e^{2t}$$

Here, we would have

$$Y = Y_1 + Y_2,$$

with Y_1 and Y_1 two particular solutions to the corresponding two sub-problems.

The general procedure should be read as

(1) Find the general solution of the corresponding homogeneous equation.
(2) Make sure that the RHS function $g(t)$ is of the form discussed above.
(3) If $g(t) = \sum_{i=1}^n g_i(t)$, split the problem into n subproblems.
(4) Take Y_i to be of approximate form with unknown coefficients, and then plug it into the equation to find the equations that the coefficients must satisfy.

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