INTRODUCTION TO DIFFERENTIAL EQUATIONS

HONGYU LIU

Lecture 2

1. First Order Differential Equations

Example 1.6. *Solve the initial value problem*

$$\begin{cases} ty' + 2y = 4t^2, \\ y(1) = 2. \end{cases} \tag{1.12}$$

<u>*Solution*</u> First, we rewrite the equation into the standard form

$$y' + (2/t)y = 4t, \tag{1.13}$$

and so $p(t) = 2/t$ and $g(t) = 4t$. We next compute the integrating factor

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln|t|} = t^2.$$

Now, the differential equation (1.13) can be equivalently transformed to

$$\left(t^2 y\right)' = 4t^3$$

and from which it follows

$$y = t^2 + \frac{c}{t^2}.$$

Finally, by using the initial condition, one can get $c = 1$, namely, the solution for the initial value problem is $y = t^2 + 1/t^2$. It is easy to verify that this solution has discontinuity at the point $t = 0$, which comes from the discontinuity of $p(t) = 2/t$ at $t = 0$. So, for our initial value problem, the solution only exists for $t > 0$. On the other hand, we would like to remark that if $c = 0$, then $y = t^2$ is a solution existing for all t. For such solution, we must impose an initial condition like $y(0) = 0$. □

1.2. **Separable Equations.** In Subsection 1.1, we have mainly considered linear equation, namely, $y' = f(t,y)$ with f dependent linearly on y. We next consider a more general subclass of first order differential equation with f not necessarily dependent linearly on y. In the following, we will use x, rather than t, to denote the independent variable. That is, we will use $y(x)$ to represent the unknown function and $y' = dy/dx$. Different letters are frequently used for the variables in a differential equation, and you should not become too accustomed to using a single pair. Now, the equation becomes

$$\frac{dy}{dx} = f(x,y), \tag{1.14}$$

which can always be rewrite (1.14) into the general form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0. \tag{1.15}$$

If it happens that M is a function of x only and N is a function of y only, then Eq. (1.15) becomes

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{1.16}$$

Such an equation is said to be **separable**, and we can write it in the differential form

$$M(x)dx + N(y)dy = 0. \tag{1.17}$$

A *separable equation* can be solved by integrating the function M and N. We illustrate the process by an example and the discuss it in general for Eq. (1.16).

Example 1.7. *Solve the equation*

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}. \tag{1.18}$$

<u>*Solution*</u> We write Eq. (1.18) as

$$-x^2 + (1-y^2)\frac{dy}{dx} = 0, \tag{1.19}$$

and so it is separable. Observe that the first term is the derivative of $-x^3/3$ and that the second term, by means of the chain rule, is the derivative w.r.t. x of $y - y^3/3$. Thus, Eq. (1.19) can be written as

$$\frac{d}{dx}(-\frac{x^3}{3}) + \frac{d}{dx}(y - \frac{y^3}{3}) = 0, \tag{1.20}$$

or

$$\frac{d}{dx}(-\frac{x^3}{3} + y - \frac{y^3}{3}) = 0. \tag{1.21}$$

Therefore, by integrating we obtain

$$-\frac{x^3}{3} + y - \frac{y^3}{3} = c, \tag{1.22}$$

where c is an arbitrary constant. In order to fix this constant, we impose for instance an initial condition, $y(0) = 1$, which then gives that $c = 2/3$. Clearly, the solution $y(x)$ to the differential equation (1.18) is implicitly implied in the equation (1.22). This is not a major obstacle, though it seems a little inconvenient. □

Essentially, the same procedure can be followed for any separable equation. Returning to Eq. (1.16), let H_1 and H_2 be any antiderivatives of M and N, respectively. That is,

$$H_1'(x) = M(x) \qquad H_2'(y) = N(y). \tag{1.23}$$

Then Eq. (1.16) becomes

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0. \tag{1.24}$$

According to the chain rule,

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dx}H_2(y). \tag{1.25}$$

Consequently, we can write Eq. (1.24) as

$$\frac{d}{dx}[H_1(x) + H_2(y)] = 0. \tag{1.26}$$

By integrating, we get

$$H_1(x) + H_2(y) = c, \tag{1.27}$$

with c an arbitrary constant. Eq. (1.27) defines the solution to (1.16) implicitly. In practice, Eq. (1.27) is usually obtained from Eq. (1.18) by integrating the first term w.r.t. x and the second term w.r.t. y. If, in addition to the differential equation, we prescribe an initial condition

$$y(x_0) = y_0. \tag{1.28}$$

Then the constant c in (1.27) is fixed by $c = H_1(x_0) + H_2(y_0)$. That is, the solution for the initial value problem (1.16) and (1.28) is given implicitly by

$$H_1(x) + H_2(y) = H_1(x_0) + H_2(y_0). \tag{1.29}$$

In order to obtain an explicit formula for the solution, it is required to solve the equation (1.29) for y as a function of x. This is usually impossible, and in such case one can resort to the numerical methods to find approximate values of y for given values of x.

Excercise 1.8. *Solve the initial value problem*

$$y' = \frac{x^2}{y(1+x^3)} \qquad y(0) = 1.$$

Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98105, USA
E-mail address: hyliu@math.washington.edu