INTRODUCTION TO DIFFERENTIAL EQUATIONS

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Lecture 8

1. First Order Differential Equations

First, continue with the Euler's method for numerically solving first order differential equation and some of its applications.

2. Second order linear differential equations

So far, we have been mainly concerned with the first order differential equation

$$y' = f(t, y).$$

Next, we turn our study to the second order differential equation of the general form

$$F(y'', y', y, t) = 0, \tag{2.1}$$

where y is a function of t and the prime represents differentiation w.r.t. t. In fact, here we could introduce a natural definition on the n-th order differential equation which should assume the following general form

$$F(y^{(n)}, y^{(n-1)}, \ldots, y'', y', y, t) = 0.$$

We would say little about the differential equations of order higher than 2.

Returning to eqn. (2.1), we say the second order equation is linear if F depends linearly on y'', y' and y'; namely,

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

Otherwise, the differential equation is called *nonlinear*. We confine our study to the linear differential equations, especially, with constant coefficients given by

$$ay'' + by' + c = g(t).$$

Here, we further call the eqn *homogeneous* if $g(t) = 0$ and *nonhomogeneous* if $g(t) \neq 0$. Later, we shall show that the nonhomogeneous equations can be solved by using sophisticated technique provided the corresponding homogeneous equation having been solved. Hence, we would first focus on how to solve the homogeneous equations. In order to gain some experience before taking up the general idea, let's consider a simple, but typical, example.

Consider the differential equation

$$y'' - y = 0.$$

Recall from the calculus, a typical function whose derivative is the same as itself is the exponential functions, e^t. A little bit more thought produce a second solution, e^{-t}. It's further verified by straightforward calculations that a linear combinations of the two functions, $c_1e^t + c_2e^{-t}$ is still a solution.

Let's turn back to the general differential equation

$$ay'' + by' + cy = 0. \tag{2.2}$$

As illustrated by the previous example, we seek a solution of the form e^{rt}. Substituting it into the differential equation, one can obtain

$$(ar^2 + br + c)e^{rt} = 0.$$

Since $e^{rt} \neq 0$, this leads to $ar^2 + br + c = 0$, which is called the *characteristic equation* for the differential equation (2.2). It's of crucial importance, since its roots provide the solution for the differential equation of the form e^{rt}. Clearly, we have two roots for the characteristic equation and the following three cases would be considered separately,

(1) The two zeros are real but different.
(2) The two zeros are real but repeated.
(3) The two zeros are complex conjugates.

For the first case, let's assume that r_1 and r_2 are two different real zeros. Clearly, $c_1e^{r_1t} + c_2e^{r_2t}$ is a solution, where c_1 and c_2 are two arbitrary constants.

Example 2.1. *Find the solution of*

$$y'' + 5y' + 6 = 0.$$

Example 2.2. *Find the solution of*

$$4y'' - 8y' + 3y = 0.$$

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