Key to Midterm

Math 307 D & E, May 2, 2008

Student No.:______________ **Name**:____________

- You have 50 minutes to complete the exam.
- There are 6 problems. Make sure that you have completed all the problems.
- A piece of notes (8:5×11 inches, letter size, one side) will be allowed on the exam, along with a scientific (but not graphing) calculator.

Problem	**Points**	**Score**
1		
2		
3		
4		
5		
6		
Total		

Problem 1. Solve the differential equation

$$\frac{dy}{dt} + \sin t \; y = e^{-\cos t} \sin t.$$

<u>**Solution:**</u> The integrating factor is

$$\mu(t) = e^{\int \sin t dt} = e^{-\cos t}.$$

Multiplying both sides of the equation by $\mu(t)$, we get

$$(\mu y)' = e^{-2\cos t} \sin t.$$

By integration, we finally have

$$y = \frac{1}{2} e^{-2\cos t} + c \; e^{\cos t}.$$

Problem 2. Find the solution of the initial value problem in explicit form

$$\frac{dy}{dx} = \frac{(2y^2+1)(x^2+\cos x)}{4y}, \quad y(0) = \frac{\sqrt{2}}{2}.$$

(You need NOT determine the validity interval of the obtained solution.)

Solution: By separating variables,

$$\frac{4y}{2y^2+1}dy = (x^3 + \cos x)dx.$$

Integrating both sides, we further have

$$\ln(2y^2+1) = \frac{x^3}{3} + \sin x + c.$$

By using the initial condition, $c = \ln 2$, and hence

$$y^2 = e^{x^3/3 + \sin x} - 1/2.$$

Again, making use of the initial condition,

$$y = \sqrt{e^{x^3/3 + \sin x} - 1/2}.$$

Problem 3. A tank initially contains 100 gal of pure water. A mixture containing a concentration of $(e^{-t/4} + 1/2)$ g/gal of chemical enters the tank at the rate of 4 gal/min, and the well-stirred mixture leaves the tank at the same rate.

(a) Construct a differential equation governing this process and find the amount of chemical in the tank at any time.

<u>**Solution:**</u> Let $Q(t)$ represent the amount of salt in the tank at time t. By the inflow/outflow principle,

$$\begin{aligned}\text{rate in} &= 4(e^{-t/4+1/2}),\\ \text{rate out} &= 4Q/100,\end{aligned}$$

and hence

$$\frac{dQ}{dt} = 4(e^{-t/4+1/2}) - 4Q/100,$$

namely,

$$\frac{dQ}{dt} = 4e^{-t/4} + 2 - Q/25. \qquad (*)$$

At time $t = 0$, there is no chemical in the tank, and hence the initial condition is $Q(0) = 0$.

Solving the obtained differential equation by using the method of integrating factor, one can get

$$Q(t) = 50 - \frac{400}{21}e^{-t/4} - \frac{650}{21}e^{-t/25}.$$

(b) Find the limiting amount of chemical in the tank.

<u>**Solution:**</u>

$$Q_L = \lim_{t\to\infty} = 50g.$$

(c) Use Euler's method with step size $h = 0.1$ to find the approximate amount of the chemical in the tank at the time $t = 0.3$.

<u>**Solution:**</u> We need to solve the differential equation (*) by using Euler's method. Letting

$$f(t, Q) := 4e^{-t/4} + 2 - Q/25,$$

the equation becomes

$$\frac{dQ}{dt} = f(t, Q).$$

Let $h = 0.1$ and $t_0 = 0, t_1 = 0.1, t_2 = 0.2, t_3$, the Euler's method is read as

$$\begin{aligned}Q_1 =& Q_0 + hf(t_0, Q_0) = 0 + 0.1 \times (e^{-0/4} + 2 - 0/25) = 0.6\\ Q_2 =& Q_1 + hf(t_1, Q_1) = 0.6 + 0.1 \times (e^{-0.1/4} + 2 - 0.6/25) \approx 1.19\\ Q_3 =& Q_2 + hf(t_2, Q_2) = 1.19 + 0.1 \times (e^{-0.2/4} + 2 - 1.19/25) \approx 1.77.\end{aligned}$$

Problem 4. Consider the differential equation

$$\frac{dy}{dt} = (y^2 - 2y + 1)(y - 2).$$

(a) Determine the equilibrium solutions of the differential equation.

Solution: Letting

$$(y^2 - 2y + 1)(y - 2) = 0,$$

one find the equilibrium solutions are

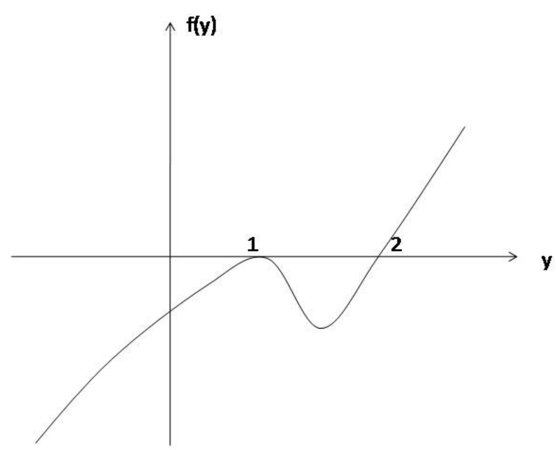
$$y_1 = 1, \quad y_2 = 2.$$

(b) Sketch the graph of several solutions in the ty-plane.

Solution: Let

$$f(y) = (y^2 - 2y + 1)(y - 2).$$

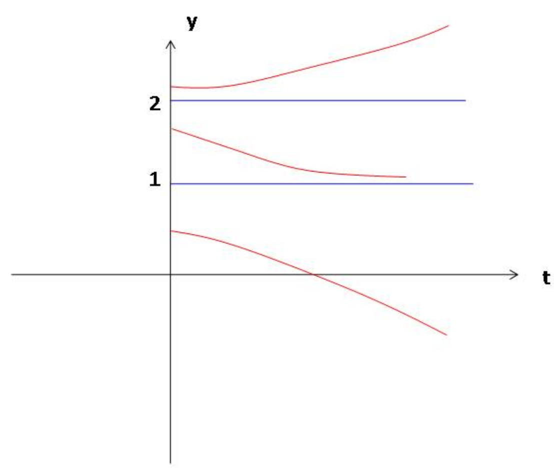
We first sketch the graph of $f(y)$ versus y as follows



From the graph, one knows that

1. $f(y) < 0$ when $y < 1$, and hence $y(t)$ is decreasing;
2. $f(y) < 0$ when $1 < y < 2$, and hence $y(t)$ is decreasing;
3. $f(y) > 0$ when $y > 2$, and hence $y(t)$ is increasing.

Now, it is easy to sketch the graph of solutions for the differential equation.



(c) Classify the equilibrium solutions as asymptotically stable or unstable.

Solution: It is easy to see from the previous graph that both equilibrium solutions are asymptotically unstable.

Problem 5. (a) Solve the following initial value problem

$$\begin{cases} y'' - 5y - +6y = 0, \\ y(0) = 5/6, \quad y'(0) = 2. \end{cases}$$

Solution: The characteristic equation is

$$r^2 - 5r + 6 = 0,$$

which gives two roots $r_1 = 2$ and $r_2 = 3$. Hence, the general solution is

$$y(t) = c_1 e^{2t} + c_2 e^{3t}.$$

Then, by using the two initial conditions, one can show that $c_1 = 1/2$ and $c_2 = 1/3$, and hence the solution for the IVP is

$$y = 1/2e^{2t} + 1/3e^{3t}.$$

(b) Find the general solution of the following differential equation

$$2y'' - 2y' + 5y = 0.$$

Solution: The characteristic equation is

$$2r^2 - 2r + 5 = 0,$$

which has two roots $r_{1,2} = 1/2 \pm 3/2i$. Hence, the general solution is

$$y = c_1 e^{t/2} \cos 3t/2 + c_2 e^{t/2} \sin 3t/2.$$

Problem 6. Consider the following differential equation

$$\alpha y'' + \beta y' + \gamma y = 0,$$

where α, β and γ are positive constants.

(a) Show that $y = 0$ is the equilibrium solution to the differential equation.

Solution: Let $y = c_0$ with c_0 being some constant is an equilibrium solution to the differential equation. Substituting it into the differential equation, one can find that

$$\alpha \times 0 + \beta \times 0 + \gamma \times c_0 = 0,$$

from which we know that $c_0 = 0$ since $\gamma > 0$. That is, if $y(t)$ is an equilibrium solution to the differential equation, then one must have $y = 0$.

(b) Show that the equilibrium solution $y = 0$ is asymptotically sate.

Solution: We first solve the differential equation. The corresponding characteristic equation is

$$\alpha r^2 + \beta r + \gamma = 0.$$

Let $\Delta := \beta^2 - 4\alpha\gamma$. We next consider three cases.

Case 1. $\Delta > 0$. In this case, the characteristic equation has two different real roots,

$$r_1 = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}, \quad r_2 = \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}.$$

Since $\alpha > 0, \beta > 0$ and $\gamma > 0$, one can easily show that

$$r_1 < 0, \quad r_2 < 0.$$

Therefore, every possible solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ and

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} (c_1 e^{r_1 t} + c_2 e^{r_2 t}) = 0.$$

Case 2. $\Delta = 0$. In this case, the roots for the characteristic equation are

$$r_1 = r_2 = -\frac{\beta}{2\alpha}.$$

Now, the general solution is

$$y = c_1 e^{-\beta t/2\alpha} + c_2 t e^{-\beta t/2\alpha}.$$

Again, one can show that

$$\lim_{t \to \infty} c_1 e^{-\beta t/2\alpha} + c_2 t e^{-\beta t/2\alpha} = 0.$$

Case 3. $\Delta < 0$. In this case, the roots for the characteristic equation are complex conjugate,

$$r_{1,2} = -\frac{\beta}{2\alpha} \pm \frac{\sqrt{4\alpha\gamma - \beta^2}}{2\alpha} i.$$

The general solution is

$$y = c_1 e^{-\beta t/2\alpha} \cos \frac{\sqrt{4\alpha\gamma - \beta^2}}{2\alpha} t + c_2 e^{-\beta t/2\alpha} \sin \frac{\sqrt{4\alpha\gamma - \beta^2}}{2\alpha} t.$$

Again, one easily see that

$$\lim_{t\to\infty} y(t) = 0.$$

In each case, we have show that $\lim_{t\to\infty} y(t) = 0$, where $y(t)$ is an arbitrary solution to the differential equation. Hence, the equilibrium solution $y = 0$ is asymptotically stable.