On Recovering Polyhedral Scatterers with Acoustic Far-field Measurements

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Abstract

We prove that an acoustic sound-hard scatterer, consisting of finitely many solid polyhedra in $\mathbb{R}^n (n \geq 2)$, is uniquely determined by the far-field patterns corresponding to $n-1$ different incident waves. By suitable modifications, the method can also be used to show a similar uniqueness result in the setting without knowing the *a priori* physical properties of the underlying scatterer.

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1 Introduction and statement of the results

We shall be concerned with the identification of hostile/inaccessible objects by using acoustic far-field measurements. Let $\mathbf{D}$ represent the underlying (unknown) scatterer, which is assumed to be a bounded Lipschitz domain in $\mathbb{R}^n (n \geq 2)$ with connected complement $\mathbf{G} := \mathbb{R}^n \backslash \bar{\mathbf{D}}$. We send an incident field $u^i = e^{\mathrm{i}kx\cdot d}$ which is a time-harmonic plane wave propagating in the direction $d \in \mathbb{S}^{n-1}$ with speed c in the homogeneous background medium, and at frequency $\omega = kc$. When meeting the scatterer $\mathbf{D}$, u^i will be perturbed and leads to a *scattered field* u^s. We further assume that the scatterer is impenetrable, that is, the scattered field only exists in the exterior domain $\mathbf{G}$. Now, we define $u := u^i + u^s$ to be the total field, then the scattering phenomenon is described by the following Helmholtz equation

$$(1.1) \qquad \begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbf{G}, \\ \lim\limits_{r\to\infty} r^{(n-1)/2}(\dfrac{\partial u^s}{\partial r} - \mathrm{i}ku^s) = 0 & r = |x|, \end{cases}$$

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where the limit holds uniformly in all directions $\hat{x} = x/|x| \in \mathbb{S}^{n-1}$ and is known as the *Sommerfeld radiation condition.* For the well-posedness of the forward scattering system, we still need to impose suitable boundary conditions on $\partial\mathbf{G}$. This is dependent on the intrinsic physical properties of the underlying scatterer. For a *sound-soft* scatterer, the pressure of the total wave vanishes on the boundary, and this gives a Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\mathbf{G}; \tag{1.2}$$

while for a *sound-hard* scatterer, the normal velocity of the acoustic wave vanishes on the boundary and we have a Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\mathbf{G} \tag{1.3}$$

where ν is the unit inward normal to $\partial\mathbf{G}$. The forward problem (1.1), (1.2) or (1.3), has been well understood, and it is known that there exists a unique solution $u \in H^1_{loc}(\mathbf{G})$. Moreover, u is (real) analytic in $\mathbf{G}$ and the asymptotic behavior of the scattered field is governed by

$$u^s(x) = \frac{e^{\mathrm{i}k|x|}}{|x|^{(n-1)/2}} \left\{ u_\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|}\right)\right\}, \tag{1.4}$$

as $|x| \to \infty$, uniformly in all directions $\hat{x} \in \mathbb{S}^{n-1}$. The function u_∞ is the so-called *far-field pattern*, and in the following we shall write $u_\infty(\hat{x}; k, d, \mathbf{D})$ to indicate its dependence on the observation direction $\hat{x}$, the wave number k, the incident direction d and the corresponding scatterer $\mathbf{D}$. We refer to [4, 12] for a more detailed study of the forward scattering problem.

By physical apparatus, one can measure the far-field data u_∞ with desired accuracy. So, from a practical viewpoint, it is of much interest to determine the unknown scatterer $\mathbf{D}$, i.e. its shape and location, by performing the far-field measurements corresponding to some given incident waves. This constitutes one of the fundamental problems in the active field of inverse scattering, and is usually referred to as *inverse acoustic obstacle scattering* (cf. [4]). Like in most of the inverse problems, the first question to ask in this context is the *identifiability*; i.e., whether a scatterer can be identified from a knowledge of its far-field patterns. We refer to [7] for a general discussion of the crucial role of uniqueness in inverse problems theory theoretically as well as numerically. Mathematically, the *identifiability* is the *uniqueness*; that is, if two scatterers $\mathbf{D}_1$ and $\mathbf{D}_2$ produce the same far-field pattern, namely $u_\infty(\hat{x}; \mathbf{D}_1) = u_\infty(\hat{x}; \mathbf{D}_2)$, can one conclude that $\mathbf{D}_1 = \mathbf{D}_2$? It is observed that the uniqueness results also provide the practical information on how many measurement data one should use to identify the scatterer. As an important ingredient in the uniqueness study and noting $u_\infty(\hat{x})$ is an analytic function, one can assume that u_∞ is only available on an open subset of the unit sphere, no matter how small the subset is, since we can always recover such data on the whole unit sphere by analytic continuation. So, for our current study, we may assume without loss of generality that $u_\infty(\hat{x})$ is given for every possible observation direction, i.e., $\hat{x} \in \mathbb{S}^{n-1}$. Now, the inverse scattering problem is formally determined with a single incident wave at arbitrarily fixed frequency and incident direction, since then the unknown $\partial\mathbf{G}$ depends on the same number of variables, $n-1$, as does the measurement data . Hence, there is a widespread belief that one can uniquely determine the scatterer by a single far-field measurement. However, this remains to be a longstanding conjecture, see for instance a recent survey paper [3] by Colton and Kress.

In the past few years, significant progress has been achieved for the unique determination of general polyhedral scatterers by minimum far-field measurements (see [1, 2, 5, 9, 10, 11]). In summary, the uniqueness for the determination of sound-soft polyhedral scatterers has been completely solved (cf. [1, 9]). In fact, it has been shown that a polyhedral scatterer in $\mathbb{R}^n$ of any dimension $n \geq 2$, possibly consisting of finitely many solid polyhedra and cracks, is uniquely determined by a single far-field measurement. Here, we define a *crack* to be an open subset of some hyperplane in $\mathbb{R}^n$. Whereas for the sound-hard case, a similar result has been established by n different far-field measurements (cf. [9]), and it is further shown to be optimal by counter examples in [10] indicating that one cannot determine uniquely a polyhedral scatterer by any less than n far-field measurements, provided it admits the simultaneous presence of crack-type components. All these results have been extended to the setting without knowing the *a priori* physical properties of the underlying scatterer, that is, it may be either sound-soft or sound-hard, or of quite complicated nature of the combinations of the two (cf. [11]). To step further, one may still anticipate the uniqueness in the sound-hard case by a single far-field measurement by excluding the presence of crack-type components, that is, the underlying scatterer is composed of solid polyhedra. This is affirmatively verified in [5, 11] for the two-dimensional problem, and the higher dimensional case is left unsolved, which is the motivation for the current study. We would like to remark that during the writing of the paper we are aware of a more recent result by Elschner and Yamamoto in [6], where they have shown the unique determination of a sound-hard polyhedral scatterer in $\mathbb{R}^n$ of any dimension $n \geq 2$ by a single far-field measurement, but under the restrictive assumption that the underlying scatterer is composed of finitely many *convex* polyhedra. The new result we are going to present will have no such geometric restriction on the scatterer other than that it is of polyhedral type. But on the other hand, we have to make use of $n-2$ more far-field measurements. We next state more precisely the main result.

It is first recalled that a (open) polyhedron in $\mathbb{R}^n$ is a simply connected set whose boundary is composed of subsets lying on $(n-1)$-dimensional hyperplanes. In the following, we call $\mathbf{D}$ a *polyhedral scatterer* if it is a bounded domain in $\mathbb{R}^n$ and consists of finitely many pairwise disjoint polyhedra. From a practical viewpoint, we further assume that the number of component polyhedra is not known *a priori*. For such general polyhedral scatterers, we shall show

Theorem 1.1. *Let* $\mathbf{D}$ *and* $\widetilde{\mathbf{D}}$ *be two polyhedral scatterers in* $\mathbb{R}^n$. *For any fixed* $k_0 > 0$, *we have* $\mathbf{D} = \widetilde{\mathbf{D}}$ *as long as*

$$u_\infty(\hat{x}; k_0, d_j, \mathbf{D}) = u_\infty(\hat{x}; k_0, d_j, \widetilde{\mathbf{D}}) \quad \text{for } \hat{x} \in \mathbb{S}^{n-1} \text{ and } j = 1, 2, \dots, n-1, \tag{1.5}$$

where $d_j \in \mathbb{S}^{n-1}$, $1 \leq j \leq n-1$ *are* $n-1$ *linearly independent incident directions.*

The method for the proof of Theorem 1.1 is not new and it follows from a similar *path argument* as that in [9], in combination with those new ingredients developed in [8] for inverse electromagnetic scattering problems, which are shown to be also applicable to the current acoustic case with necessary modifications. Nonetheless, we would like to present the uniqueness result for two purposes. On the one hand, for dimension $n = 2$, Theorem 1.1 obviously recovers the uniqueness results in [5] and [11] by a single far-field measurement. But the argument for the proof of Theorem 1.1 is basically different from those implemented in [5, 11], and moreover it is simplified considerably especially compared to that in [11]. On the other hand, for the three dimensional case, we have made use of two different far-field measurements, and this obviously improves earlier results in literature. But it is remarked that the uniqueness with formally determined data of a single far-field measurement in this three dimensional case still remains unsolved. In the following, we would frequently refer to [8] and [9] for similar technical details.

2 Proof of Theorem 1.1

In order to stick closely to [9], we need to modify the definitions of Neumann set and Neumann hyperplanes to the current case by using $n-1$ incident waves (cf. Def. 1, [9]). We start by fixing some notations for the subsequent use. Let $B_r(x)$ denote an open ball in $\mathbb{R}^n$ with center x and radius r, and $\bar{B}_r(x)$ and $S_r(x)$, respectively, the closure and boundary of $B_r(x)$. The notation $T_r(x)$ is defined to be an open cube of edge length r, centered at x. Unless specified otherwise, ν shall always denote the inward normal to a concerned domain, or the one-sided normal to some hyperplane. Henceforth, we fix the wave number $k_0 > 0$, and the $n-1$ linearly independent incident directions $d_j, 1 \leq j \leq n-1$, as those stated in Theorem 1.1. We write $\mathscr{U}(x; \mathbf{D}) := \{u(x; d_j, \mathbf{D})\}_{j=1}^{n-1}$, and the operation on $\mathscr{U}(x)$ is understood to be elementwise.

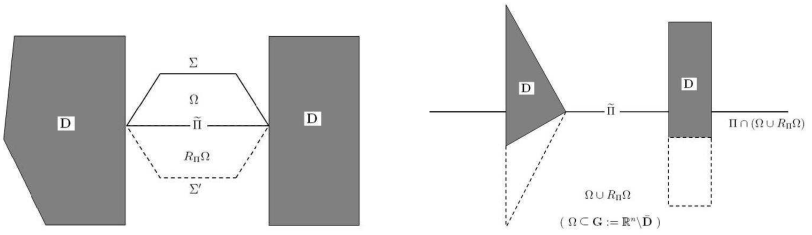

Figure 1: Two dimensional illustration to Theorem 2.2, the left one is for case (i) and the right one is for case (ii) .

Definition 2.1. *$\mathscr{N}_{\mathscr{U}}$ is called the Neumann set of $\mathscr{U}$ in $\mathbf{G} := \mathbb{R}^n \backslash \bar{\mathbf{D}}$ if*

$$\mathscr{N}_{\mathscr{U}} = \{x \in \mathbf{G};\ \partial_\nu \mathscr{U}|_{\Pi \cap B_r(x) \cap \mathbf{G}} = 0 \text{ for some } r > 0 \text{ and hyperplane } \Pi \text{ passing through } x\}. \tag{2.1}$$

For any $x \in \mathscr{N}_{\mathscr{U}}$, we let Π be the hyperplane involved in the definition of $\mathscr{N}_{\mathscr{U}}$ and $\widetilde{\Pi}$ be the connected component of $\Pi \backslash \mathbf{D}$ containing x. By analytical continuation, we see $\partial_\nu \mathscr{U} = 0$ on $\widetilde{\Pi}$. In the sequel, such $\widetilde{\Pi}$ will be referred to as a *Neumann hyperplane*. A very useful property of Neumann hyperplanes is the so-called *reflection principle*, which is summarized in the following theorem (cf. [9]). Starting from now on, we denote by R_Π the reflection in $\mathbb{R}^n$ with respect to a hyperplane Π.

Theorem 2.2. *For a connected polyhedral domain $\Omega \subset \mathbf{G}$, let $\widetilde{\Pi}$ be one of its faces that lies on some Neumann hyperplane. Furthermore, let Π be the hyperplane in $\mathbb{R}^n$ containing $\widetilde{\Pi}$ and $\Omega \cup R_\Pi \Omega \subset \mathbf{G}$. We have two consequences (see Fig. 1 for a two dimensional illustration):*

(i) $\partial_{\nu_\Pi} \mathscr{U} = 0$ *on* $\Pi \cap (\Omega \cup R_\Pi \Omega)$;

(ii) Suppose that $\Sigma \subset \partial\Omega$ is a subset of one face of Ω other than $\widetilde{\Pi}$ and $\partial \nu_\Sigma \mathscr{U} = 0$ on Σ, then we have $\partial_{\nu_{\Sigma'}} \mathscr{U} = 0$ on $\Sigma' := R_\Pi \Sigma$.

It is verified directly that there might exist unbounded Neumann hyperplanes. This constitutes one of the major differences from those Neumann hyperplanes introduced in [9]. All the Neumann hyperplanes defined there are bounded due to the use of n different incident waves, see Lemma 2 in [9]. With the reflection principle in Theorem 2.2, we proceed to classify all the Neumann hyperplanes into two sets, one is bounded and the other is unbounded. To this end, we fix a Neumann hyperplane $\widetilde{\Pi}$ for our subsequent discussion and let Π be the hyperlane in $\mathbb{R}^n$ containing $\widetilde{\Pi}$. We further localize our investigation by fixing a point $x_0 \in \widetilde{\Pi} \cap \mathbf{G}$ and take a sufficiently small ball $B_{\tau_0}(x_0) \subset \mathbf{G}$. $B_{\tau_0}(x_0)$ is divided by $\widetilde{\Pi}$ into two half balls, which we respectively denote by B^+ and B^-. Let $\mathbf{G}^\pm$ be respectively the connected components of $\mathbf{G} \backslash \widetilde{\Pi}$ containing $\mathrm{B}^\pm$. Next, let $\boldsymbol{\Lambda}^\pm$ be respectively

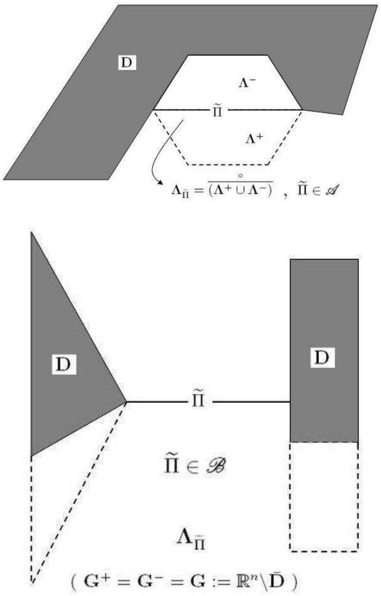

Figure 2: Two dimensional illustration to the sets $\mathscr{A}$ and $\mathscr{B}$, the upper one is for $\mathscr{A}$ and the lower one is for $\mathscr{B}$.

the connected components of $\mathbf{G}^\pm \cap R_\Pi(\mathbf{G}^\mp)$ containing $\mathrm{B}^\pm$. Finally, set $\mathbf{\Lambda} = \overset{\circ}{\overline{(\mathbf{\Lambda}^+ \cup \mathbf{\Lambda}^-)}}$ and we see that $\mathbf{\Lambda}$ is a polyhedral domain which is symmetric with respect to Π and $B_{\tau_0}(x_0) \subset \mathbf{\Lambda}$. By the reflection principle in (ii) of Theorem 2.2, we know $\partial_\nu \mathscr{U} = 0$ on $\partial\mathbf{\Lambda}$. It is observed that the construction of $\mathbf{\Lambda}$ is independent of the choice of x_0 and τ_0 and is only dependent on $\widetilde{\Pi}$. In the following, we shall write $\mathbf{\Lambda}_{\widetilde{\Pi}}$ to denote the symmetric set constructed as above corresponding to some Neumann hyperplane $\widetilde{\Pi}$. Now, we set (see Fig. 2 for a two-dimensional illustration)

$$\mathscr{A} = \{\widetilde{\Pi};\ \widetilde{\Pi} \text{ is a Neumann hyperplane with bounded } \mathbf{\Lambda}_{\widetilde{\Pi}}\}; \tag{2.2}$$

$$\mathscr{B} = \{\widetilde{\Pi};\ \widetilde{\Pi} \text{ is a Neumann hyperplane with unbounded } \mathbf{\Lambda}_{\widetilde{\Pi}}\}. \tag{2.3}$$

It is remarked that if $\widetilde{\Pi}$ is an unbounded Neumann hyperplane, one always has $\widetilde{\Pi} \in \mathscr{B}$. In fact, in such cases, one can verify directly that the corresponding $\mathbf{\Lambda}_{\widetilde{\Pi}}$ would contain the exterior of a sufficiently large ball containing $\mathbf{D}$. We next present two crucial lemmata on the set $\mathscr{B}$.

Lemma 2.3. *There exists a hyperplane* Π *in* $\mathbb{R}^n$ *such that* $\mathscr{B} \subset \Pi$.

Proof. Let $\widetilde{\Pi}_\kappa \in \mathscr{B}$, $\kappa = 1, 2$ be two Neumann hyperplanes with respective normals ν_κ, $\kappa = 1, 2$ and let Π_κ denote the hyperplane in $\mathbb{R}^n$ containing $\widetilde{\Pi}_\kappa$. By using the reflection principle (i) in Theorem 2.2, we see that $\partial_{\nu_\kappa} \mathscr{U} = 0$ on

$\Pi_\kappa \cap \mathbf{\Lambda}_{\widetilde{\Pi}_\kappa}$ for $\kappa = 1, 2$. Noting that $\mathbf{\Lambda}_{\widetilde{\Pi}_\kappa}$ contains the exterior of a sufficiently large ball containing $\mathbf{D}$, we may assume without loss of generality that both $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$ are unbounded. Next, by using the fact that $\lim_{|x|\to\infty} |\nabla \mathscr{U}^s(x)| = 0$, one can derive by straightforward calculations that $\nu_\kappa \cdot d_l = 0$ for $\kappa = 1, 2$ and $l = 1, 2, \ldots, n-1$. Since $d_l \in \mathbb{S}^{n-1}$, $1 \le l \le n-1$ are linearly independent, this in turn implies that $\nu_1 \parallel \nu_2$, namely, $\widetilde{\Pi}_1 \parallel \widetilde{\Pi}_2$. Clearly, we now only need to show that $\Pi_1 = \Pi_2 := \Pi$. Assume contrarily that $\Pi_1 \neq \Pi_2$, and hence $\tau_0 = dist(\Pi_1, \Pi_2) > 0$. We next let $T_r(0)$ be a cube with sufficiently large $r > 0$ such that $\mathbf{D} \subset T_r(0)$, and by suitable rotation, we assume that $\widetilde{\Pi}_\kappa, \kappa = 1, 2$ are perpendicular to one face of $T_r(0)$. Set $\Sigma_\kappa = \widetilde{\Pi}_\kappa \backslash T_r(0)$ for $\kappa = 1, 2$. We now see by the reflection principle (ii) in Theorem 2.2 that $\partial_\nu \mathscr{U} = 0$ on $\Sigma_{l+2} = R_{\Pi_{l+1}}(\Sigma_l)$, $l = 1, 2, \ldots$, where Π_l is the hyperplane in $\mathbb{R}^n$ containing Σ_l. Clearly, we have $dist(\Sigma_l, \Sigma_{l+1}) = \tau_0$ for $l = 1, 2, \ldots$. Hence, there must exist $l_0 < \infty$ such that $\mathbf{D}$ lies entirely on one side of Π_{l_0}. Again using Theorem 2.2, we make the reflection of $\partial\mathbf{D}$ with respect to Π_{l_0} and then find two non-parallel unbounded Neumann hyperplanes which are extended from the two adjacent faces of $R_{\Pi_{l_0}}(\partial\mathbf{D})$. This contradicts to our earlier conclusion derived in the first part of the present proof that two unbounded Neumann hyperplanes must be parallel to each other and thus proving the lemma. □

Lemma 2.4. *The open set* $\mathbf{G}\backslash\bar{\mathscr{B}}$ *has no bounded connected component.*

Proof. The proof follows from a similar argument as that for Lemma 2.7 in [8]. □

Now, we are in a position to present the proof of the main theorem.

Proof of Theorem 1.1. We assume contrarily that $\mathbf{D} \neq \widetilde{\mathbf{D}}$. Let Ω be the (unique) unbounded connected component of $\mathbb{R}^n\backslash\overline{(\mathbf{D} \cup \widetilde{\mathbf{D}})}$. Since $\mathbf{D} \neq \widetilde{\mathbf{D}}$, we may without loss of generality assume that $(\mathbb{R}^n\backslash\bar{\Omega})\backslash\bar{\mathbf{D}} \neq \emptyset$ and let D^* be a connected component of this non-empty set. Clearly, $D^* \subset \widetilde{\mathbf{D}}$ and hence is bounded. By Rellich's theorem (cf. Theorem 6.9, [4]), we infer from (1.5) that $\mathscr{U}(x; \mathbf{D}) = \mathscr{U}(x; \widetilde{\mathbf{D}})$ in Ω. This, together with the fact $\partial D^* \subset \partial\Omega \cup \bar{\mathbf{D}} \subset \partial\mathbf{D} \cup \partial\widetilde{\mathbf{D}}$, implies by using the homogeneous Neumann boundary conditions of $\mathscr{U}(x; \mathbf{D})$ and $\mathscr{U}(x; \widetilde{\mathbf{D}})$ on $\partial\mathbf{D}$ and $\partial\widetilde{\mathbf{D}}$ that $\partial_\nu \mathscr{U}(x; \mathbf{D}) = 0$ on ∂D^*. With such preparation, the proof follows basically from a similar *path argument* as that in [9]. But those new ingredients developed in [8] for inverse electromagnetic scattering problems must be implemented here for the current acoustic scattering problem by using Lemmata 2.3 and 2.4. Hence in the following, we only sketch the main steps for the proof. Starting from now on, we shall simply write $\mathscr{U}$ to denote $\mathscr{U}(x; \mathbf{D})$, and $\widetilde{\Pi}_l$ with an integer index l to denote a Neumann hyperplane while Π_l is the hyperplane in $\mathbb{R}^n$ containing $\widetilde{\Pi}_l$.

Step I: Existence of a Neumann hyperplane $\widetilde{\Pi}_1 \in \mathscr{A}$

It is first noted that $\partial D^*\backslash\bar{\mathbf{D}}\neq\emptyset$. Hence, there must be an open face say Γ_0 on ∂D^* that can be extended in $\mathbf{G}$ to form a Neumann hyperplane, which we denote by $\widetilde{\Pi}_0$. If $\widetilde{\Pi}_0\in\mathscr{A}$, then we are done. So, without loss of generality, we assume that $\widetilde{\Pi}_0\in\mathscr{B}$. Let Θ be the connected component of $(D^*\cup\widetilde{\Pi}_0\cup R_{\Pi_0}D^*)\cap\mathbf{\Lambda}_{\widetilde{\Pi}_0}$ containing Σ_0. Since D^* is bounded, we know Θ is bounded. Moreover noting $\partial\Theta\subset\partial\mathbf{D}\cup R_{\Pi_0}(\partial\mathbf{D})$, by the reflection principle (ii) of Theorem 2.2, $\partial_\nu\mathscr{U}=0$ on $\partial\Theta$. Obviously, $\partial\Theta\backslash\bar{\mathbf{D}}\neq\emptyset$. Therefore, there must be an open face Γ_1 lying on $\partial\Theta\backslash\partial\mathbf{D}$ which can be extended in $\mathbf{G}$ to form a Neumann hyperplane $\widetilde{\Pi}_1$. Since Θ is symmetric with respect to Π_0, we know $\widetilde{\Pi}_1$ is non-coplanar to $\widetilde{\Pi}_0$. Noting $\widetilde{\Pi}_0\in\mathscr{B}$ and by Lemma 2.4, $\widetilde{\Pi}_1\in\mathscr{A}$.

Step II: Construction of an exit path γ

We let Q be the (unique) unbounded connected component of $\mathbf{G}\backslash\widetilde{\Pi}_1$. Clearly, $\widetilde{\Pi}_1\subset\partial Q$ and by noting both $\partial\mathbf{G}$ and $\widetilde{\Pi}_1$ are bounded, Q contains the exterior of a sufficiently large ball containing $\mathbf{D}$. Fix an arbitrary point $x_1\in\widetilde{\Pi}_1$ and let $\gamma:=\gamma(t)(t\geq 0)$ be a regular curve such that $\gamma(t_1)=x_1$ with $t_1=0$, and $\gamma(t)\subset Q$ and $\lim_{t\to\infty}|\gamma(t)|=\infty$. Next, if $\gamma\cap\mathscr{B}\neq\emptyset$, we would modify γ as follows to require that γ has only one intersection point with $\mathscr{B}$. In fact, we set $x^*=\gamma(t^*)$ be the 'first' intersection point of $\gamma(t)(t>0)$ with $\mathscr{B}$; that is, $t^*=\min\{t>0;\gamma(t)\in\mathscr{B}\}<\infty$. Let W be the connected component of $\mathbf{G}\backslash\mathscr{B}$ such that $x^*\in\partial W$. Then, we let $O:=Q\cap W$ and it is straightforward to verify that Q is an unbounded open connected set such that $x^*\in\partial O$. In fact, the connectedness of Q is obvious by noting that both Q and W are connected. Whereas the unboundedness of O is due to the facts that W is unbounded by Lemma 2.4 and Q contains the exterior of a sufficiently large ball as mentioned earlier. Next, let $\eta(t)(t\geq t^*)$ be a regular curve such that $\eta(t^*)=x^*$, $\eta(t)\subset O$ for $t>t^*$ and $\lim_{t\to\infty}|\eta(t)|=\infty$. Furthermore, it is trivial to require that $\eta(t)$ has C^1-connection with $\gamma(t)(0\leq t\leq t^*)$ at x^*. Now, set $\tilde{\gamma}(t)$ be $\gamma(t)$ for $t\in[0,t^*]$ and $\eta(t)$ for $t>t^*$, then $\tilde{\gamma}(t)(t\geq 0)$ satisfies our requirements of an exit path.

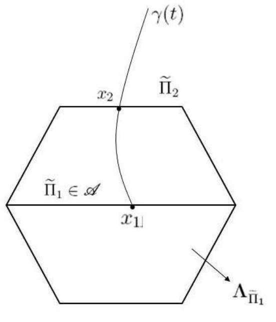

Figure 3: Two-dimensional illustration to Step III in the proof of Theorem 1.1.

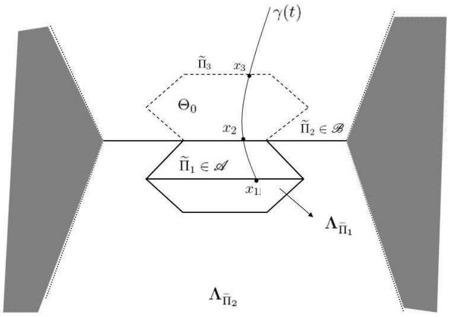

Figure 4: Two-dimensional illustration to Step III in the proof of Theorem 1.1 in the case that $\widetilde{\Pi}_2 \in \mathscr{B}$.

Step III: Continuation of Neumann hyperplanes along γ

Let $d_0 = dist(\gamma, \mathbf{D}) > 0$ and $r_0 = d_0/2$. Since $\widetilde{\Pi}_1 \in \mathscr{A}$, we know the corresponding symmetric set $\mathbf{\Lambda}_{\widetilde{\Pi}_1}$ is bounded. Hence, $\gamma \cap \partial\mathbf{\Lambda}_{\widetilde{\Pi}_1} \neq \emptyset$. Let $x_2 = \gamma(t_2)$ be the 'last' intersection point of γ with $\partial\mathbf{\Lambda}_{\widetilde{\Pi}_1}$; namely, $t_2 = \max\{t > 0; \gamma(t) \in \partial\mathbf{\Lambda}_{\widetilde{\Pi}_1}\}$, and this then implies the existence of a Neumann hyperplane $\widetilde{\Pi}_2$ passing though x_2 which is extended from an open of $\partial\mathbf{\Lambda}_{\widetilde{\Pi}_1}$ in $\mathbf{G}$ (see Fig. 3 for a two-dimensional illustration). Moreover, by a standard technique of *path argument* (cf. the proof of Theorem 1 in [9]), we know $|\gamma(t_1 \leq t \leq t_2)| > r_0$. Next, if $\widetilde{\Pi}_2 \in \mathscr{A}$, we may repeat the above argument to find a third Neumann hyperplane $\widetilde{\Pi}_3$ and $x_3 := \gamma(t_3) \in \widetilde{\Pi}_3$ such that $|\gamma(t_2 \leq t \leq t_3)| > r_0$. But on the other hand, it may happen that $\widetilde{\Pi}_2 \in \mathscr{B}$. For this latter case, we let Θ_0 be the connected component of $(\mathbf{\Lambda}_{\widetilde{\Pi}_1} \cup \widetilde{\Pi}_2 \cup R_{\Pi_2}(\mathbf{\Lambda}_{\widetilde{\Pi}_1})) \cap \mathbf{\Lambda}_{\widetilde{\Pi}_2}$ containing x_2 (see Fig. 4 for a two-dimensional illustration). Since $\mathbf{\Lambda}_{\widetilde{\Pi}_1}$ is bounded, we know Θ_0 is bounded. Then it is straightforward to verify that $\gamma \cap \partial\Theta_0 \neq \emptyset$ and we let $x_3 := \gamma(t_3)$ be the 'last' intersection point of γ with $\partial\Theta_0$. Clearly, $\partial\Theta_0 \subset \partial\mathbf{D} \cup R_{\Pi_0}(\partial\mathbf{D})$, by the reflection principle we have $\partial_\nu \mathscr{U} = 0$ on $\partial\Theta_0$, and this further implies the existence of a Neumann hyperplane $\widetilde{\Pi}_3$ which is extended from an open face $\partial\Theta_0$ in $\mathbf{G}$. Then, noting Θ_0 is symmetric with respect to Π_2 and $\widetilde{\Pi}_2 \in \mathscr{B}$, we must have $\widetilde{\Pi}_3 \in \mathscr{A}$. It is remarked that we would only have $|\gamma(t_2 \leq t \leq t_3)| > 0$ but not $> r_0$ in this latter case with $\widetilde{\Pi}_2 \in \mathscr{B}$ (see the proof of Theorem 1.1 in [8] for similar discussion). According to our construction of γ in Step II, we know that $\gamma(t)$ and $\mathscr{B}$ have at most one intersection point. Hence, we can continue with the above procedure to find a sequence of points $x_n := \gamma(t_n), n = 1, 2, \ldots$ such that $\gamma(t_n) \in \widetilde{\Pi}_n$. Furthermore, there exists an $l_0 < \infty$ such that when $n > l_0$, $\widetilde{\Pi}_n \in \mathscr{A}$ and $|\gamma(t_n \leq t \leq t_{n+1})| > r_0$. Here, the index l_0 corresponds to the case that $\widetilde{\Pi}_{l_0} \in \mathscr{B}$ and it may happen that $l_0 = 0$. Now, we can conclude our proof by a typical contradiction in *path argument*. First, it is clear that $\mathscr{A} \subset \overline{ch(\mathbf{D})}$, where $ch(\mathbf{D})$ is the convex hull of $\mathbf{D}$ and is obviously bounded. Hence, there exits $T < \infty$ such that $\lim_{n\to} t_n = T$. In turn, we have $\lim_{n\to\infty} |\gamma(t_n \leq t \leq t_{n+1})| = \int_{t_n}^{t_{n+1}} |\gamma'(t)|\, dt = 0$, contradicting our construction that $|\gamma(t_n \leq t_{n+1})| > r_0$ when $n > l_0$ and thus completing the proof. $\square$

3 Uniqueness for mixed type scatterers

In this last section, we remark that the uniqueness in Theorem 2.2 holds also for mixed type scatterers as described in [11]. In fact, the scatterer may also admit the simultaneous presence of sound-soft crack-type components. Next, we first recall the definition of a mixed type polyhedral scatterer. Let $\mathbf{D}$ be a bounded domain with connected complement $\mathbf{G}$ and $\mathbf{D} = (\bigcup_{l=1}^{m_1} D_l) \cup (\bigcup_{l=1}^{m_2} C_l)$, where D_l's are pairwise disjoint polyhedra and C_l's are cracks. On the boundary of $\mathbf{D}$ we consider the following mixed boundary conditions for the forward scattering problem (1.1),

$$u = 0 \quad \text{on} \quad \Gamma_D \cup (\bigcup_{l=1}^{m_2} C_l) \quad \text{and} \quad \partial_\nu u = 0 \quad \text{on} \quad \Gamma_N,$$

where Γ_D and Γ_N are a Lipschitz dissection of $\cup_{l=1}^{m_1} \partial D_l$. We refer to such a $\mathbf{D}$ as *a mixed type polyhedral scatterer*. Then, Theorem 1.1 is still valid when both $\mathbf{D}$ and $\widetilde{\mathbf{D}}$ are replaced by mixed type polyhedral scatterers. For the proof, we can introduce the so-called Dirichlet hyperplane as that in Definition 2 in [9]. It is known that the reflection principle given in Theorem 2.2 holds also for such mixed type scatterers (cf. [11]). Moreover, by Lemma 5 in [9], we know that all the Dirichlet hyperplanes are bounded. Hence we would encounter the least difficulty in treating Dirichlet hyperplanes. Consequently, we can prove the uniqueness in such a setting without the *a priori* physical properties of the underlying scatterers by a similar argument as what we have done for solely sound-hard scatterers.

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